

Mirror Symmetry for Stable Quotients Invariants

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Abstract

The moduli space of stable quotients introduced by Marian-Oprea-Pandharipande provides a natural compactification of the space of morphisms from nonsingular curves to a nonsingular variety. When the latter is a Grassmannian, the moduli space of stable quotients carries a canonical virtual class. We show that the analogue of Givental's J -function for the resulting twisted projective invariants is described by the same mirror hypergeometric series as the corresponding Gromov-Witten invariants (which arise from the moduli space of stable maps), but without the mirror transform (in the Calabi-Yau case). This implies that the stable quotients and Gromov-Witten twisted invariants agree if there is enough "positivity", but not in all cases. As a corollary of the proof, we show that certain twisted Hurwitz numbers arising in the stable quotients theory are also described by a fundamental object associated with this hypergeometric series. We thus completely answer some of the questions posed by Marian-Oprea-Pandharipande concerning their invariants. Our results suggest a deep connection between the stable quotients invariants of complete intersections and the geometry of the mirror families.

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1 Introduction

Gromov-Witten invariants of a smooth projective variety X are certain counts of curves in X that arise from integrating against the virtual class of the moduli space of stable maps. These are known to possess striking structures which are often completely unexpected from the classical point of view. For example, the genus 0 Gromov-Witten invariants of a quintic threefold, i.e. a degree 5 hypersurface in \mathbb{P}^4 , are related by a so-called **mirror formula** to a certain hypergeometric series. This relation was explicitly predicted in [2] and mathematically confirmed in [5] and [11] in the 1990s. In fact, the prediction of [2] has been shown to be a special case of mirror symmetry for certain twisted Gromov-Witten invariants of projective complete intersections of sufficiently small total multi-degree ([6], [10]); these invariants are associated with direct sums of line bundles (positive and negative) over \mathbb{P}^n . This relation is often described by assembling two-point Gromov-Witten invariants (but without constraints on the second marked point) into a generating function, known as Givental's J -function. In most cases (in particular, when the anticanonical class of the corresponding complete intersection is at least twice the hyperplane class), the J -function precisely equals to the appropriate hypergeometric series. In certain borderline cases, they differ by a simple exponential factor. In the remaining Calabi-Yau cases, the correcting factors are more complicated and the two power series also differ by a change of the power series variable, known as the mirror transform. The situation for stable quotients invariants turns out to be much simpler.

The moduli spaces of stable quotients, $\overline{Q}_{g,m}(X, d)$, constructed in [13], provide an alternative to the moduli spaces of stable maps, $\overline{M}_{g,m}(X, d)$, for compactifying spaces of degree d morphisms from genus g nonsingular curves with m marked points to a projective variety X (with a choice of polarization).¹ The space $\overline{Q}_{g,m}(\mathbb{P}^{n-1}, d)$ consists of equivalence classes of tuples

$$(\mathcal{C}, y_1, \dots, y_m, S \subset \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{C}}),$$

where $(\mathcal{C}, y_1, \dots, y_m)$ is a genus g nodal curve with m marked points and $S \subset \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{C}}$ is a subsheaf of rank 1 and degree $-d$, that satisfy certain stability and torsion properties; see Section 2. This moduli space is smooth if $g=0$ or $(g, m)=(1, 0)$ and carries a virtual class in all cases. There is a natural surjective contraction morphism

$$c: \overline{M}_{g,m}(X, d) \longrightarrow \overline{Q}_{g,m}(X, d),$$

which is not injective for $d > 0$ and generally contracts a lot of boundary strata. For example, $\overline{Q}_{1,0}(\mathbb{P}^{n-1}, d)$ is irreducible and has Picard rank just 2; see [3, Theorem 4.1]. Thus, the moduli spaces of stable quotients are much more efficient compactifications than the moduli spaces of stable maps.

As in the case of stable maps, there are evaluation morphisms,

$$\text{ev}_i: \overline{Q}_{g,m}(\mathbb{P}^{n-1}, d) \longrightarrow \mathbb{P}^{n-1}, \quad i = 1, 2, \dots, m,$$

corresponding to each marked point.² There is also a universal curve

$$\pi: \mathcal{U} \longrightarrow \overline{Q}_{g,m}(\mathbb{P}^{n-1}, d)$$

¹These “compactifications”, $\overline{Q}_{g,m}(X, d)$ and $\overline{M}_{g,m}(X, d)$, are generally just compact spaces containing the spaces of morphisms; the latter need not be dense in $\overline{Q}_{g,m}(X, d)$ and $\overline{M}_{g,m}(X, d)$.

²The morphism ev_i sends a tuple $(\mathcal{C}, y_1, \dots, y_m, S)$ to the line $S_x \subset \mathbb{C}^n$ if S is viewed as a line subbundle of the trivial rank n bundle over \mathcal{C} .

with m sections $\sigma_1, \dots, \sigma_m$ (given by the marked points) and a universal rank 1 subsheaf

$$\mathcal{S} \subset \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{U}}.$$

For each $i=1, 2, \dots, m$, let

$$\psi_i = -\pi_*(\sigma_i^2) \in H^2(\overline{Q}_{g,m}(\mathbb{P}^{n-1}, d))$$

be the first chern class of the universal cotangent line bundle as usual. By [13, Theorems 2,3], the moduli space $\overline{Q}_{g,m}(\mathbb{P}^{n-1}, d)$ carries a canonical virtual class and

$$c_*[\overline{\mathfrak{M}}_{g,m}(\mathbb{P}^{n-1}, d)]^{\text{vir}} = [\overline{Q}_{g,m}(\mathbb{P}^{n-1}, d)]^{\text{vir}}. \quad (1.1)$$

Since the evaluation morphisms ev_i and the ψ -classes on the two moduli spaces commute with c and c^* , respectively, (1.1) implies that the (untwisted) Gromov-Witten and stable quotients invariants of \mathbb{P}^{n-1} , obtained by integrating pull-backs of cohomology classes on \mathbb{P}^{n-1} by ev_i and powers of ψ -classes against the two virtual classes, are the same; see [13, Theorem 3]. In this paper, we study twisted invariants in genus 0, arising from sums of line bundles over \mathbb{P}^{n-1} ; they relate invariants of projective complete intersections to the invariants of ambient space.

For $l \in \mathbb{Z}^{\geq 0}$ and l -tuple $\mathbf{a} = (a_1, \dots, a_l) \in (\mathbb{Z}^*)^l$ of nonzero integers, let

$$\begin{aligned} |\mathbf{a}| &= \sum_{k=1}^l |a_k|, & \langle \mathbf{a} \rangle &= \prod_{a_k > 0} a_k / \prod_{a_k < 0} a_k, & \mathbf{a}! &= \prod_{a_k > 0} a_k!, & \mathbf{a}^{\mathbf{a}} &= \prod_{k=1}^l a_k^{|a_k|}, \\ \ell^{\pm}(\mathbf{a}) &= |\{k: (\pm 1)a_k > 0\}|, & \ell(\mathbf{a}) &= \ell^+(\mathbf{a}) - \ell^-(\mathbf{a}). \end{aligned}$$

If in addition $n \in \mathbb{Z}^+$ and $d \in \mathbb{Z}^+$, let

$$\mathcal{V}_{n;\mathbf{a}}^{(d)} = \bigoplus_{a_k > 0} R^0 \pi_*(\mathcal{S}^{*a_k}(-\sigma_1)) \oplus \bigoplus_{a_k < 0} R^1 \pi_*(\mathcal{S}^{*a_k}(-\sigma_1)) \longrightarrow \overline{Q}_{0,2}(\mathbb{P}^{n-1}, d), \quad (1.2)$$

where $\pi: \mathcal{U} \longrightarrow \overline{Q}_{0,2}(\mathbb{P}^{n-1}, d)$ is the universal curve; this sheaf is locally free. The euler class of the analogue of this sheaf in Gromov-Witten theory describes the genus 0 invariants of the total space of the vector bundle

$$\bigoplus_{a_k < 0} \mathcal{O}_{\mathbb{P}^{n-1}}(a_k) \Big|_{X_{(a_k)_{a_k > 0}}} \longrightarrow X_{(a_k)_{a_k > 0}}, \quad (1.3)$$

where $X_{(a_k)_{a_k > 0}} \subset \mathbb{P}^{n-1}$ is a nonsingular complete intersection of multi-degree $(a_k)_{a_k > 0}$. The stable quotients analogue of Givental's J -function is given by

$$Z_{n;\mathbf{a}}(x, \hbar, q) \equiv 1 + \sum_{d=1}^{\infty} q^d \text{ev}_{1*} \left[\frac{e(\mathcal{V}_{n;\mathbf{a}}^{(d)})}{\hbar - \psi_1} \right] \in H^*(\mathbb{P}^{n-1})[[\hbar^{-1}, q]], \quad (1.4)$$

where $\text{ev}_1: \overline{Q}_{0,2}(\mathbb{P}^{n-1}, d) \longrightarrow \mathbb{P}^{n-1}$ is as before and $x \in H^2(\mathbb{P}^{n-1})$ denotes the hyperplane class.

The hypergeometric series describing Givental's J -function in Gromov-Witten theory is given by

$$Y_{n;\mathbf{a}}(x, \hbar, q) \equiv \sum_{d=0}^{\infty} q^d \frac{\prod_{a_k > 0} \prod_{r=1}^{a_k d} (a_k x + r \hbar) \prod_{a_k < 0} \prod_{r=0}^{-a_k d-1} (a_k x - r \hbar)}{\prod_{r=1}^d (x + r \hbar)^n} \in \mathbb{Q}[x][[\hbar^{-1}, q]]. \quad (1.5)$$

In the pure Calabi-Yau case, i.e. $a_k > 0$ for all k and $|\mathbf{a}| = n$, we also need the power series

$$I_{n;\mathbf{a}}(q) \equiv Y_{n;\mathbf{a}}(0, 1, q) = \begin{cases} 1, & \text{if } |\mathbf{a}| - |\ell^-(\mathbf{a})| < n; \\ \sum_{d=0}^{\infty} q^d \frac{\prod_{k=1}^l (a_k d)!}{(d!)^n}, & \text{if } |\mathbf{a}| - |\ell^-(\mathbf{a})| = n. \end{cases}$$

By the following theorem, the stable quotients analogue of Givental's J -function is also described by the hypergeometric series (1.5), but in a more straightforward way.

Theorem 1. *If $l \in \mathbb{Z}^{\geq 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$ are such that $|\mathbf{a}| \leq n$, then the stable quotients analogue of Givental's J -function satisfies*

$$Z_{n;\mathbf{a}}(x, \hbar, q) = \frac{Y_{n;\mathbf{a}}(x, \hbar, q)}{I_{n;\mathbf{a}}(q)} \in H^*(\mathbb{P}^{n-1})[[\hbar^{-1}, q]]. \quad (1.6)$$

If $|\mathbf{a}| - \ell^-(\mathbf{a}) \leq n - 2$, this is the same identity as for Givental's J -function; see [6, Theorem 9.1] for the $\ell^-(\mathbf{a}) = 0$ case and [4, Theorem 5.1] for the $\ell^-(\mathbf{a}) \geq 1$ case. Thus, the Gromov-Witten invariants and stable quotients invariants agree if $|\mathbf{a}| - \ell^-(\mathbf{a}) \leq n - 2$. If $|\mathbf{a}| = n - 1$ and $\ell^-(\mathbf{a}) = 0$, the Gromov-Witten analogue of (1.6) is the relation

$$Z_{n;\mathbf{a}}^{\text{GW}}(x, \hbar, q) = e^{-\mathbf{a}!q/\hbar} \frac{Y_{n;\mathbf{a}}(x, \hbar, q)}{I_{n;\mathbf{a}}(q)};$$

see [6, Theorem 10.7]. Finally, if $|\mathbf{a}| = n$ and $\ell^-(\mathbf{a}) \leq 1$, the Gromov-Witten analogue of (1.6) involves a mirror transform between the power series variable on the left-hand side (now denoted by Q) and the power series variable q on the right-hand side. It takes the form

$$Z_{n;\mathbf{a}}^{\text{GW}}(x, \hbar, Q) = e^{-J(q)x/\hbar} \frac{Y_{n;\mathbf{a}}(x, \hbar, q)}{I_{n;\mathbf{a}}(q)}, \quad \text{where } Q = q \cdot e^{J_{n;\mathbf{a}}(q)},$$

for a certain a power series $J_{n;\mathbf{a}}(q) \in q \cdot \mathbb{Q}[[q]]$; see [6, Theorem 11.8] for the $\ell^-(\mathbf{a}) = 0$ case and [4, Theorem 5.1] for the $\ell^-(\mathbf{a}) = 1$ case. The same comparison applies to the equivariant version of Theorem 1, Theorem 3 in Section 4, and its Gromov-Witten analogues, [6, Theorems 9.5, 10.7, 11.8] in the $\ell^-(\mathbf{a}) = 0$ case and [4, Theorem 5.3] in the $\ell^-(\mathbf{a}) \geq 1$ case.

As in the case of mirror symmetry for Gromov-Witten invariants, Theorem 1 follows immediately from its \mathbb{T}^n -equivariant version, Theorem 3 in Section 4. The latter is proved using the Atiyah-Bott localization theorem [1] on $\overline{Q}_{0,2}(\mathbb{P}^{n-1}, d)$, which reduces the equivariant version of the power series (1.4), the power series $\mathcal{Z}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q)$ defined by (4.1) below, to a sum of rational functions over certain graphs. As in the case of Gromov-Witten invariants, $\mathcal{Z}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q)$ is \mathfrak{C} -recursive in the sense of Definition 5.1, with the collection \mathfrak{C} of structure coefficients given by (5.6), and satisfies the self-polynomiality condition of Definition 5.2; the same is the case of the equivariant version of the power series (1.5), the power series $\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q)$ defined by (4.2). Thus, the two power series

$$\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q), \mathcal{Z}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q) \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[[\hbar^{-1}, q]]$$

are determined by their mod $(\hbar^{-1})^2$ part; see Proposition 5.3. It is straightforward to determine the mod $(\hbar^{-1})^2$ -part of the power series $\mathcal{Y}_{n;\mathbf{a}}$. The mod $(\hbar^{-1})^2$ -part of Givental's J -function in

Gromov-Witten theory is 1 in all cases for a simple geometric reason. This approach thus confirms the analogue of Theorem 1 in Gromov-Witten theory and thus mirror symmetry for the genus 0 Gromov-Witten invariants of projective complete intersections.

For stable quotients, the situation with the mod $(\hbar^{-1})^2$ -part of $\mathcal{Z}_{n;\mathbf{a}}$ is different. It is still 1, for dimensional reasons, if $|\mathbf{a}| \leq n-2$. If $|\mathbf{a}| = n-1$, the mod $(\hbar^{-1})^2$ -part of $\mathcal{Z}_{n;\mathbf{a}}$ vanishes in the q -degrees 2 and higher; it is straightforward to see that the coefficient of q^1 mod $(\hbar^{-1})^2$ is $\mathbf{a}!/\hbar$ if $\ell^-(\mathbf{a})=0$ and 0 otherwise.³ So, in these cases, the proof of mirror symmetry for Gromov-Witten invariants carries over to the stable quotients invariants. However, in the Calabi-Yau case, $|\mathbf{a}|=n$, the mod $(\hbar^{-1})^2$ -part of $\mathcal{Z}_{n;\mathbf{a}}$ is not zero in all q -degrees if $\ell^-(\mathbf{a}) \leq 1$, and we see no a priori reason for the coefficients of positive q -degrees to vanish even if $\ell^-(\mathbf{a}) \geq 2$. Thus, the proof of mirror symmetry for Gromov-Witten invariants cannot directly carry over to the stable quotients invariants in the Calabi-Yau cases.

Since the coefficients of q^0 on the two sides of the identity in Theorem 3 are the same (both are 1), it is equivalent to the equality of the auxiliary coefficients, $\mathcal{Y}_i^r(d)$ and $\mathcal{Z}_i^r(d)$, in the recursions (5.4) for $\mathcal{Y}_{n;\mathbf{a}}$ and $\mathcal{Z}_{n;\mathbf{a}}$, respectively. By a direct algebraic computation, the coefficients $\mathcal{Y}_i^r(d)$ are expressible in terms of certain residues of \mathcal{Y} ; see Lemma 5.4. Analyzing the relevant graphs, one can show that the coefficients $\mathcal{Z}_i^r(d)$ are likewise expressible in terms of certain residues of \mathcal{Z} , but in a different way; see Proposition 6.1. Thus, for each pair (n, \mathbf{a}) with $|\mathbf{a}| \leq n$, the identity in Theorem 3 is equivalent to certain identities for the residues of $\mathcal{Y}_{n;\mathbf{a}}$; see Lemma 8.2. Since $\mathcal{Y}_{n;\mathbf{a}} = \mathcal{Z}_{n;\mathbf{a}}$ whenever $|\mathbf{a}| \leq n-2$, these identities hold whenever $|\mathbf{a}| \leq n-2$. On the other hand, the validity of these identities is independent of n , and so they thus hold for all pairs (n, \mathbf{a}) ; see Proposition 8.3. This yields Theorem 3 and thus Theorem 1.

The relations of Lemma 8.2 involve twisted Hurwitz numbers arising from certain moduli spaces of weighted stable curves $\overline{\mathcal{M}}_{0,2|d}$; see Section 2. These relations in turn uniquely determine the twisted Hurwitz numbers, even equivariantly, in terms of a key power series associated with $\mathcal{Y}_{\mathbf{a};n}$; see Theorems 2 and 4 in Sections 2 and 4, respectively. Based on developments in Gromov-Witten theory, one would expect these closed formulas to be a key ingredient in computing twisted genus 1 stable quotients invariants and thus answering yet another question raised in [13].

The proof that the equivariant version of Givental's J -function in Gromov-Witten theory satisfies the self-polynomiality condition of Definition 5.2 uses the localization theorem [1] to compute integrals over the moduli space $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^{n-1}, (1, d))$. Our proof that the equivariant stable quotients analogue of Givental's J -function satisfies the self-polynomiality condition uses the moduli space of stable pairs of quotients $\overline{\mathcal{Q}}_{0,2}(\mathbb{P}^1 \times \mathbb{P}^{n-1}, (1, d))$ in a similar way; see Section 7. This moduli space is a special case of moduli space

$$\overline{\mathcal{Q}}_{g,m}(\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_p-1}, (d_1, \dots, d_p))$$

of stable p -tuples of quotients, which we describe in Section 2 by extending the notion of stable quotients introduced in [13].

The Gromov-Witten analogues of Theorem 1 and its equivariant version, Theorem 3 in Section 4, extend to the so-called **concave bundles** over products of projective spaces, i.e. vector bundles of

³Even this is not necessary due to our approach to the Calabi-Yau case.

the form

$$\bigoplus_{k=1}^l \mathcal{O}_{\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_p-1}}(a_{k;1}, \dots, a_{k;p}) \longrightarrow \mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_p-1},$$

where for each given $k = 1, 2, \dots, l$ either $a_{k;1}, \dots, a_{k;p} \in \mathbb{Z}^{\geq 0}$ or $a_{k;1}, \dots, a_{k;p} \in \mathbb{Z}^-$. The stable quotients analogue of these bundles are the sheaves

$$\bigoplus_{k=1}^l \mathcal{S}_1^{*a_{k;1}} \otimes \dots \otimes \mathcal{S}_p^{*a_{k;p}} \longrightarrow \mathcal{U} \longrightarrow \overline{Q}_{0,2}(\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_p-1}, (d_1, \dots, d_p)) \quad (1.7)$$

with the same condition on $a_{k;i}$, where $\mathcal{S}_i \longrightarrow \mathcal{U}$ is the universal subsheaf corresponding to the i -th factor; see Section 2. In this case, we compare two power series

$$Y_{n_1, \dots, n_p; \mathbf{a}}(x_1, \dots, x_p, \hbar, q_1, \dots, q_p) \in \mathbb{Q}[x_1, \dots, x_p][[\hbar^{-1}, q_1, \dots, q_p]], \quad (1.8)$$

$$Z_{n_1, \dots, n_p; \mathbf{a}}(x_1, \dots, x_p, \hbar, q_1, \dots, q_p) \in H^*(\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_p-1})[[\hbar^{-1}, q_1, \dots, q_p]], \quad (1.9)$$

where $x_1, \dots, x_p \in H^*(\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_p-1})$ are the pullbacks of the hyperplane classes by the projection maps. The coefficient of $q_1^{d_1} \dots q_p^{d_p}$ in (1.9) is defined by the same pushforward as in (1.4), with the degree d of the stable maps replaced by (d_1, \dots, d_p) . The coefficient of $q_1^{d_1} \dots q_p^{d_p}$ in (1.8) is obtained from the coefficients in (1.5) by replacing $a_k d$ and $a_k x$ by $a_{k;1} d_1 + \dots + a_{k;p} d_p$ and $a_{k;1} x_1 + \dots + a_{k;p} x_p$ in the numerator and taking the product of the denominators with $(n, x, d) = (n_s, x_s, d_s)$ for each $s = 1, \dots, p$. In the condition $|\mathbf{a}| \leq n$, $|\mathbf{a}|$ is still the sum of the absolute values of all $a_{i,j}$, while $n = n_1 + \dots + n_p$. Our proof of Theorem 3 (and thus of Theorem 1) extends directly to this situation; we will comment on the necessary modifications in each step of the proof.

Mirror formulas for the two-pointed versions of (1.4) and (4.1), i.e. with ev_1 and $(\hbar - \psi_1)$ replaced by $\text{ev}_1 \times \text{ev}_2$ and $(\hbar_1 - \psi_1)(\hbar_2 - \psi_2)$, as well as their generalizations to products of projective spaces, can now be readily obtained using the approaches of [17] and [14] in Gromov-Witten theory. They are related to the corresponding formulas in Gromov-Witten theory in the same ways as the one-pointed formulas; see the paragraph following Theorem 1. Based on developments in Gromov-Witten theory, one would expect such two-pointed genus 0 formulas to be useful for computing twisted genus 1 stable quotients invariants.

A notable feature of the mirror formula of Theorem 1 and its two-point analogue is that they are invariant under replacing $(n, (a_1, \dots, a_k))$ by $(n+1, (a_1, \dots, a_k, 1))$; their extensions to products of projective spaces have a similar feature.⁴ This suggests that whenever $\ell^-(\mathbf{a}) = 0$ the corresponding twisted invariants are in fact invariants of a nonsingular complete intersection $X_{\mathbf{a}} \subset \mathbb{P}^{n-1}$, viewed as a projective variety. Based on the situation in Gromov-Witten theory, one would expect that these twisted invariants of \mathbb{P}^{n-1} equal to the (untwisted) invariants of $X_{\mathbf{a}}$ obtained by integrating against the virtual class of the moduli space $\overline{Q}_{0,2}(X_{\mathbf{a}}, d)$ of stable quotients with values in X ; see [13, Section 10.1]. However, it is not yet known if this moduli space carries a virtual class.

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⁴This replacement does not change the total space of the vector bundle (1.3)

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2 Moduli spaces of stable quotients

We begin this section by extending the notion of stable quotients introduced in [13] to stable tuples of quotients. The key results of [13] readily generalize to moduli spaces of these objects. We then introduce related moduli spaces of weighted curves and give a partial description of intersections of natural classes on these spaces in Corollary 2.5; this is used in the proof of Theorems 1 and 3 in Section 8. We conclude this section with a closed formula for twisted Hurwitz numbers arising from integrals over these moduli spaces of curves; see Theorem 2.

By a **nodal genus g curve**, we will mean a reduced connected scheme \mathcal{C} over \mathbb{C} of pure dimension 1 with at worst nodal singularities and $h^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) = g$. Let $\mathcal{C}^* \subset \mathcal{C}$ denote the nonsingular locus of such a curve. A **quasi-stable genus g m -marked curve** is a tuple $(\mathcal{C}, y_1, \dots, y_m)$ consisting of a nodal genus g curve and distinct points $y_i \in \mathcal{C}^*$. A **quasi-stable quotient of the trivial rank n sheaf** on such a curve is a subsheaf $S \subset \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{C}}$ such that the support of the torsion $\tau(Q)$ of the corresponding quotient sheaf Q ,

$$0 \longrightarrow S \longrightarrow \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{C}} \longrightarrow Q \longrightarrow 0,$$

is contained in $\mathcal{C}^* - \{y_1, \dots, y_m\}$. A tuple (S_1, \dots, S_p) of quasi-stable quotients on $(\mathcal{C}, y_1, \dots, y_m)$ is **stable** if the \mathbb{Q} -line bundle

$$\omega_{\mathcal{C}}(y_1 + \dots + y_m) \otimes (\Lambda^{\text{top}} S_1^*)^{\epsilon} \otimes \dots \otimes (\Lambda^{\text{top}} S_p^*)^{\epsilon} \longrightarrow \mathcal{C}$$

is ample for all $\epsilon \in \mathbb{Q}^+$; this implies that $2g - 2 + m \geq 0$. An **isomorphism**

$$\phi: (\mathcal{C}, y_1, \dots, y_m, S_1, \dots, S_p) \longrightarrow (\mathcal{C}', y'_1, \dots, y'_m, S'_1, \dots, S'_p)$$

between tuples of quasi-stable quotients is an isomorphism $\phi: \mathcal{C} \longrightarrow \mathcal{C}'$ such that

$$\phi(y_i) = y'_i \quad \forall i = 1, \dots, m, \quad \phi^* S'_j = S_j \subset \mathbb{C}^{n_j} \otimes \mathcal{O}_{\mathcal{C}} \quad \forall j = 1, \dots, p.$$

The automorphism group of any stable tuple of quotients is finite.

Proposition 2.1. *The moduli space*

$$\overline{\mathcal{Q}}_{g,m}(\mathbb{G}(r_1, n_1) \times \dots \times \mathbb{G}(r_p, n_p), (d_1, \dots, d_p)) \tag{2.1}$$

parameterizing the stable p -tuples of quotients

$$(\mathcal{C}, y_1, \dots, y_m, S_1, \dots, S_p), \tag{2.2}$$

with $h^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) = g$, $S_i \subset \mathbb{C}^{n_i} \otimes \mathcal{O}_{\mathcal{C}}$, $\text{rk}(S_i) = r_i$, and $\deg(S_i) = -d_i$, is a separated and proper Deligne-Mumford stack of finite type over \mathbb{C} and carries a canonical two-term obstruction theory.

Proof. The construction of $\overline{Q}_{g,m}(\mathbb{G}(r,n),d)$ in [13] carries through with minor changes. We sketch the modification here.

I. *Construction of the moduli space.* Let g, m, d_1, \dots, d_p satisfy

$$2g-2+m+\epsilon(d_1+\dots+d_p) > 0 \quad \forall \epsilon > 0.$$

Let $d = d_1 + \dots + d_p$. Fix a stable p -tuple of quotients $(\mathcal{C}, y_1, \dots, y_m, S_1, \dots, S_p)$, where

$$0 \longrightarrow S_i \longrightarrow \mathbb{C}^{n_i} \otimes \mathcal{O}_{\mathcal{C}} \longrightarrow Q_i \longrightarrow 0. \quad (2.3)$$

By assumption, the line bundle

$$\mathcal{L}_{\epsilon} = \omega_{\mathcal{C}}(y_1 + \dots + y_m) \otimes (\Lambda^{r_1} S_1^* \otimes \dots \otimes \Lambda^{r_p} S_p^*)^{\epsilon}$$

is ample for all $\epsilon > 0$. Fix $\epsilon = 1/(d+1)$ and let $f = 5(d+1)$. By [13, Lemma 5], the line bundle \mathcal{L}_{ϵ}^f is very ample and has no higher cohomology. Therefore,

$$h^0(\mathcal{C}, \mathcal{L}_{\epsilon}^f) = 1 - g + 5(d+1)(2g-2+m) + 5d$$

is independent of the choice of the stable p -tuple of quotients. Let

$$V = H^0(\mathcal{C}, \mathcal{L}_{\epsilon}^f)^*.$$

The line bundle \mathcal{L}_{ϵ}^f induces an embedding $\iota: \mathcal{C} \hookrightarrow \mathbb{P}(V)$. Let Hilb denote the Hilbert scheme of curves in $\mathbb{P}(V)$ of genus g and degree

$$5(d+1)(2g-2+m) + 5d = \deg \mathcal{L}_{\epsilon}^f.$$

Each stable quotient gives rise to a point in

$$\mathcal{H} = \text{Hilb} \times \mathbb{P}(V)^m,$$

where the last factors record the locations of the markings y_1, \dots, y_m .

Points in \mathcal{H} correspond to tuples $(\mathcal{C}, y_1, \dots, y_m)$. Denote by $\mathcal{H}' \subset \mathcal{H}$ the quasi-projective subscheme consisting of the tuples such that

- (i) the points y_1, \dots, y_m are contained in \mathcal{C} ,
- (ii) the curve $(\mathcal{C}, y_1, \dots, y_m)$ is quasi-stable.

Let $\pi: \mathcal{U}' \longrightarrow \mathcal{H}'$ be the universal curve over \mathcal{H}' . For $i=1, \dots, p$, let

$$\text{Quot}(n_i - r_i, d_i) \longrightarrow \mathcal{H}',$$

be the π -relative Quot scheme parameterizing rank $(n_i - r_i)$ degree d_i quotients

$$0 \longrightarrow S_i \longrightarrow \mathbb{C}^{n_i} \otimes \mathcal{O}_{\mathcal{C}} \longrightarrow Q_i \longrightarrow 0$$

on the fibers of π . Denote by \mathcal{Q} be the fiber product

$$\mathcal{Q} = \text{Quot}(n_1 - r_1, d_1) \times_{\mathcal{H}'} \dots \times_{\mathcal{H}'} \text{Quot}(n_p - r_p, d_p) \longrightarrow \mathcal{H}'$$

and $\mathcal{Q}' \subset \mathcal{Q}$ the locally closed subscheme consisting of the tuples such that

- (iii) Q_i is locally free at the nodes and markings of \mathcal{C} ,
- (iv) the restriction of $\mathcal{O}_{\mathbb{P}(V)}(1)$ to \mathcal{C} agrees with the line bundle

$$(\omega(y_1 + \dots + y_m))^{5(d+1)} \otimes (\Lambda^{r_1} S_1^* \otimes \dots \otimes \Lambda^{r_p} S_p^*)^5.$$

The action of $\mathrm{PGL}(V)$ on \mathcal{H} extends to \mathcal{H}' and \mathcal{Q}' . A $\mathrm{PGL}(V)$ -orbit in \mathcal{Q}' corresponds to a stable quotient up to isomorphism. By stability, each orbit has finite stabilizers. The moduli space (2.1) is the stack quotient $[\mathcal{Q}'/\mathrm{PGL}(V)]$.

II. *Separateness.* We prove that the moduli stack (2.1) is separated by the valuative criterion. Let $(\Delta, 0)$ be a nonsingular pointed curve and $\Delta^0 = \Delta - \{0\}$. Take two flat families of quasi-stable pointed curves

$$\mathcal{X}^j \longrightarrow \Delta, \quad y_1^j, \dots, y_m^j: \Delta \longrightarrow \mathcal{X}^j,$$

and two flat families of stable quotients

$$0 \longrightarrow S_i^j \longrightarrow \mathbb{C}^{n_i} \otimes \mathcal{O}_{\mathcal{X}^j} \longrightarrow Q_i^j \longrightarrow 0,$$

with $j=1,2$ and $i=1, \dots, p$. Assume the two families are isomorphic away from the central fiber. The isomorphism in fact extends over 0 because by [13, Section 6.2], it extends to the families of curves $\mathcal{X}^j \longrightarrow \Delta$ in a manner preserving the sections and hence extends to each pair of families of stable quotients.

III. *Properness.* We prove the moduli stack (2.1) is proper, again by the valuative criterion. Let

$$\pi^0: \mathcal{X}^0 \longrightarrow \Delta^0, \quad y_1, \dots, y_m: \Delta^0 \longrightarrow \mathcal{X}^0$$

carry a flat family of stable p -tuples of quotients

$$0 \longrightarrow S_i \longrightarrow \mathbb{C}^{n_i} \otimes \mathcal{O}_{\mathcal{X}^0} \longrightarrow Q_i \longrightarrow 0.$$

By [13, Section 6.3], each stable quotient individually extends, possibly after base-change, and hence the p -tuple extends.

IV. *Obstruction Theory.* We follow the argument in [13, Section 3.2]. Let $\phi: \mathcal{C} \longrightarrow \mathcal{M}_{g,m}$ be the universal curve over the Artin stack of pointed curves and $\mathbf{Q}(r, n, d) \longrightarrow \mathcal{M}_{g,m}$ the relative Quot scheme of rank $n-r$ degree d quotients of $\mathbb{C}^n \otimes \mathcal{O}_{\mathcal{C}}$ along the fibers of ϕ . Denote by

$$\mathbf{Q}'(r, n, d) \subset \mathbf{Q}(r, n, d)$$

the locus consisting of locally free subsheaves and by

$$\nu: \mathbf{Q}' \equiv \mathbf{Q}'_{(r_1, n_1, d_1)} \times_{\mathcal{M}_{g,m}} \dots \times_{\mathcal{M}_{g,m}} \mathbf{Q}'(r_p, n_p, d_p) \times_{\mathcal{M}_{g,m}} \mathcal{C} \longrightarrow \mathcal{M}_{g,m}$$

the fiber product. The universal sequence of sheaves

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{C}} \longrightarrow \mathcal{Q} \longrightarrow 0$$

over $\mathbf{Q}'(r, n, d) \times_{\mathcal{M}_{g,m}} \mathcal{C}$ gives rise to a universal sequence

$$0 \longrightarrow \bigoplus_{i=1}^p \mathcal{S}_i \longrightarrow \bigoplus_{i=1}^p (\mathbb{C}^{n_i} \otimes \mathcal{O}_{\mathcal{C}}) \longrightarrow \bigoplus_{i=1}^p \mathcal{Q}_i \longrightarrow 0$$

over $\mathbf{Q}' \times_{\mathcal{M}_{g,m}} \mathcal{C}$. Let $\pi: \mathbf{Q}' \times_{\mathcal{M}_{g,m}} \mathcal{C} \longrightarrow \mathbf{Q}'$ be the projection map. By [15, Prop 4.4.4] with

$$\mathcal{K} = \bigoplus_{i=1}^p \mathcal{S}_i, \quad \mathcal{H} = \bigoplus_{i=1}^p \mathbb{C}^{n_i} \otimes \mathcal{O}_{\mathcal{C}}, \quad \text{and} \quad \mathcal{F} = \bigoplus_{i=1}^p \mathcal{Q}_i,$$

the relative deformation-obstruction theory of $\nu: \mathbf{Q}' \longrightarrow \mathcal{M}_{g,m}$ is given by

$$RHom_{\pi}(\mathcal{S}_1, \mathcal{Q}_1) \oplus \dots \oplus RHom_{\pi}(\mathcal{S}_p, \mathcal{Q}_p) = \bigoplus_{i=1}^p R\pi_* Hom(\mathcal{S}_i, \mathcal{Q}_i);$$

the equality above holds because each \mathcal{S}_i is a locally free sheaf. By [12, Section 2], $R\pi_* Hom(\mathcal{S}_i, \mathcal{Q}_i)$ can be resolved by a two-step complex of vector bundles. Thus, we conclude that

$$\nu^A: \overline{\mathcal{Q}}_{g,m}(\mathbb{G}(r_1, n_1) \times \dots \times \mathbb{G}(r_p, n_p), (d_1, \dots, d_p)) \longrightarrow \mathcal{M}_{g,m}$$

admits a two-term relative deformation-obstruction theory. Along with the smoothness of $\mathcal{M}_{g,m}$, this induces an absolute two-term deformation-obstruction theory of the moduli space (2.1); see [7, Appendix B]. \square

Proposition 2.2. *If $g=0$ or $(g, m) = (1, 0)$ and $d_1, \dots, d_p \geq 1$, then the moduli space (2.1) is a nonsingular irreducible Deligne-Mumford stack of the expected dimension.*

Proof. By part IV in the proof of Proposition 2.1, the moduli space (2.1) is smooth at a point $(\mathcal{C}, y_1, \dots, y_m, S_1, \dots, S_p)$ if

$$\bigoplus_{i=1}^p \text{Ext}^1(\mathcal{S}_i, \mathcal{Q}_i) = 0. \quad (2.4)$$

Since each \mathcal{S}_i is locally free, this is the case if

$$H^1(\mathcal{S}_i^* \otimes \mathcal{Q}_i) = 0 \quad (2.5)$$

for each $i=1, \dots, p$. From the cohomology long exact sequence for the short exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathbb{C}^{n_i} \otimes \mathcal{S}_i^* \longrightarrow \mathcal{Q}_i \otimes \mathcal{S}_i^* \longrightarrow 0,$$

we see that (2.5) holds if $H^1(\mathcal{S}_i^*) = 0$.

If $g=0$, \mathcal{C} is a rational curve and thus there are no special line bundles on \mathcal{C} that have a nonnegative degree on every component of \mathcal{C} . If $(g, m) = (1, 0)$, then \mathcal{C} is either a nonsingular curve of genus 1 or a cycle of rational curves; thus, there are no special line bundles of positive degree on \mathcal{C} that have nonnegative degree on each component of \mathcal{C} . In either case, we conclude that $H^1(\mathcal{S}_i^*) = 0$ for each $i=1, \dots, p$ and so (2.4) holds.

Thus, the moduli space (2.1) is smooth at every point and hence is a nonsingular Deligne-Mumford stack of the expected dimension. It is irreducible because it contains a dense open subset isomorphic to an open subset of an irreducible bundle over the Deligne-Mumford moduli space of stable genus g marked curves $\overline{\mathcal{M}}_{g, m+d_1+\dots+d_p}$. \square

A stable tuple as in (2.2) such that each quotient sheaf $Q_i = \mathbb{C}^{n_i} \otimes \mathcal{O}_C / S_i$ is torsion-free corresponds to a stable morphism

$$\mathcal{C} \longrightarrow \mathbb{G}(r_1, n_1) \times \dots \times \mathbb{G}(r_p, n_p)$$

with marked points y_1, \dots, y_m . As in the $p = 1$ case considered in [13, Section 3.1], there are evaluation morphisms

$$\text{ev}_i: \overline{Q}_{g,m}(\mathbb{G}(r_1, n_1) \times \dots \times \mathbb{G}(r_p, n_p), (d_1, \dots, d_p)) \longrightarrow \mathbb{G}(r_1, n_1) \times \dots \times \mathbb{G}(r_p, n_p)$$

with $i = 1, 2, \dots, m$. There is also a universal curve

$$\pi: \mathcal{U} \longrightarrow \overline{Q}_{g,m}(\mathbb{G}(r_1, n_1) \times \dots \times \mathbb{G}(r_p, n_p), (d_1, \dots, d_p))$$

with m sections $\sigma_1, \dots, \sigma_m$ and universal rank r_i subsheaves $\mathcal{S}_i \subset \mathbb{C}^{n_i} \otimes \mathcal{O}_{\mathcal{U}}$.

We will also need a certain moduli space of weighted curves; this is the stable quotients counterpart of the Deligne-Mumford moduli space of stable genus g marked curves in Gromov-Witten theory. A d -tuple of flecks on a quasi-stable m -marked curve $(\mathcal{C}, y_1, \dots, y_m)$ is a d -tuple $(\hat{y}_1, \dots, \hat{y}_d)$ of points of $\mathcal{C}^* - \{y_1, \dots, y_m\}$. Such a tuple is **stable** if the \mathbb{Q} -line bundle

$$\omega_{\mathcal{C}}(y_1 + \dots + y_m + \epsilon(\hat{y}_1 + \dots + \hat{y}_d)) \longrightarrow \mathcal{C}$$

is ample for all $\epsilon \in \mathbb{Q}^+$; this again implies that $2g - 2 + m \geq 0$. An isomorphism

$$\phi: (\mathcal{C}, y_1, \dots, y_m, \hat{y}_1, \dots, \hat{y}_d) \longrightarrow (\mathcal{C}', y'_1, \dots, y'_m, \hat{y}'_1, \dots, \hat{y}'_d)$$

between curves with m marked points and d flecks is an isomorphism $\phi: \mathcal{C} \longrightarrow \mathcal{C}'$ such that

$$\phi(y_i) = y'_i \quad \forall i = 1, \dots, m, \quad \phi(\hat{y}_j) = \hat{y}'_j \quad \forall j = 1, \dots, d.$$

The automorphism group of any stable curve with m marked points and d flecks is finite.

Proposition 2.3. *If $g, m, d \in \mathbb{Z}^{\geq 0}$, the moduli space $\overline{\mathcal{M}}_{g,m|d}$ parameterizing the stable genus g curves with m marked points and d flecks,*

$$(\mathcal{C}, y_1, \dots, y_m, \hat{y}_1, \dots, \hat{y}_d), \tag{2.6}$$

is a nonsingular, irreducible, proper Deligne-Mumford stack.

Proof. The moduli space $\overline{\mathcal{M}}_{g,m|d}$ is the moduli space of weighted pointed stable curves, defined in [8, Section 2], with m points of weight 1 and d points of weight $1/d$ (if $d > 0$). Thus, this proposition is a special case of [8, Theorem 2.1]. \square

Any tuple as in (2.6) induces a quasi-stable quotient

$$\mathcal{O}_{\mathcal{C}}(-\hat{y}_1 - \dots - \hat{y}_d) \subset \mathcal{O}_{\mathcal{C}} \equiv \mathbb{C}^1 \otimes \mathcal{O}_{\mathcal{C}}.$$

For any ordered partition $d = d_1 + \dots + d_p$ with $d_1, \dots, d_p \in \mathbb{Z}^{\geq 0}$, this correspondence gives rise to a morphism

$$\overline{\mathcal{M}}_{g,m|d} \longrightarrow \overline{Q}_{g,m}(\mathbb{G}(1, 1) \times \dots \times \mathbb{G}(1, 1), (d_1, \dots, d_p)).$$

In turn, this morphism induces an isomorphism

$$\phi: \overline{\mathcal{M}}_{g,m|d}/\mathbb{S}_{d_1} \times \dots \times \mathbb{S}_{d_p} \xrightarrow{\sim} \overline{\mathcal{Q}}_{g,m}(\mathbb{G}(1,1) \times \dots \times \mathbb{G}(1,1), (d_1, \dots, d_p)), \quad (2.7)$$

with the symmetric group \mathbb{S}_{d_1} acting on $\overline{\mathcal{M}}_{g,m|d}$ by permuting the points $\hat{y}_1, \dots, \hat{y}_{d_1}$, \mathbb{S}_{d_2} acting on $\overline{\mathcal{M}}_{g,m|d}$ by permuting the points $\hat{y}_{d_1+1}, \dots, \hat{y}_{d_1+d_2}$, etc.

There is again a universal curve

$$\pi: \mathcal{U} \longrightarrow \overline{\mathcal{M}}_{g,m|d}$$

with sections $\sigma_1, \dots, \sigma_m$ and $\hat{\sigma}_1, \dots, \hat{\sigma}_d$. Let

$$\psi_i = -\pi_*(\sigma_i^2), \quad \hat{\psi}_j = -\pi_*(\hat{\sigma}_j^2) \in H^2(\overline{\mathcal{M}}_{g,m|d}) \quad (2.8)$$

be the first chern classes of the universal cotangent line bundles.

For any subset $J \subset \{1, \dots, d\}$ with $|J| \geq 2$, denote by

$$\Delta_J \in H^*(\overline{\mathcal{M}}_{g,m|d}) \quad (2.9)$$

the class of the “diagonal”

$$\{[\mathcal{C}, y_1, \dots, y_m, \hat{y}_1, \dots, \hat{y}_d] : \hat{y}_{j_1} = \hat{y}_{j_2} \quad \forall j_1, j_2 \in J\}.$$

Let $H'(\overline{\mathcal{M}}_{g,m|d}) \subset H^*(\overline{\mathcal{M}}_{g,m|d})$ be the subring generated by the classes $\psi_i, \hat{\psi}_j, \Delta_J$, with all possible choices of i, j , and J . Denote by $\tilde{H}^*(\overline{\mathcal{M}}_{g,m|d})$ the image of $H^*(\overline{\mathcal{M}}_{g,m|d})$ in $(H'(\overline{\mathcal{M}}_{g,m|d}))^*$ under the Poincare pairing,

$$\int_{\overline{\mathcal{M}}_{g,m|d}} : H^*(\overline{\mathcal{M}}_{g,m|d}) \longrightarrow (H'(\overline{\mathcal{M}}_{g,m|d}))^*.$$

By Poincare duality, $\tilde{H}^*(\overline{\mathcal{M}}_{g,m|d}) = (H'(\overline{\mathcal{M}}_{g,m|d}))^*$, which is not relevant for our purposes, and $\tilde{H}^*(\overline{\mathcal{M}}_{g,m|d})$ is a quotient of $H^*(\overline{\mathcal{M}}_{g,m|d})$.

We show in the proof of Proposition 8.3 that the euler classes of the bundles $\mathcal{V}'_{\mathbf{a};d}$ defined in (2.15) below are linear combinations of products of the diagonal classes Δ_J and the fleck ψ -classes $\hat{\psi}_i$ and thus lie in $H'(\overline{\mathcal{M}}_{g,m|d})$. We will need to integrate products of these euler classes against the usual ψ -classes ψ_i . Thus, we need to understand only the images of these euler classes in $\tilde{H}^*(\overline{\mathcal{M}}_{g,m|d})$, the dual of $H'(\overline{\mathcal{M}}_{g,m|d})$. This is facilitated by Corollary 2.5 below.

Lemma 2.4 ([13, Section 4.5]). *If $d \in \mathbb{Z}^+$ and $a_1, a_2, b_1, \dots, b_d \in \mathbb{Z}^{\geq 0}$, then*

$$\int_{\overline{\mathcal{M}}_{0,2|d}} \psi_1^{a_1} \psi_2^{a_2} \hat{\psi}_1^{b_1} \dots \hat{\psi}_d^{b_d} = \binom{d-1}{a_1, a_2} \cdot \begin{cases} 1, & \text{if } b_1, \dots, b_d = 0 \quad \forall i, j; \\ 0, & \text{otherwise.} \end{cases} \quad (2.10)$$

Proof. If $d > 1$, there is a forgetful morphism

$$f: \overline{\mathcal{M}}_{0,2|d} \longrightarrow \overline{\mathcal{M}}_{0,2|d-1},$$

dropping the fleck \hat{y}_d . For $i=1, 2$, let $D_i \subset \overline{\mathcal{M}}_{0,2|d}$ denote the divisor whose generic element consists of two components, with one of them containing y_i and \hat{y}_d (and no other marked points). By (2.8),

$$\psi_i = f^* \psi_i + D_i \quad \forall i = 1, 2, \quad \hat{\psi}_j = f^* \psi_j \quad \forall j = 1, \dots, d-1. \quad (2.11)$$

Under the canonical identification of $D_i \approx \overline{\mathcal{M}}_{0,2|d-1} \times \overline{\mathcal{M}}_{0,2|1}$ with $\overline{\mathcal{M}}_{0,2|d-1}$,

$$\begin{aligned} D_i|_{D_i} &= -\psi_i, & D_1 \cdot D_2 &= 0, & \psi_i|_{D_i}, \hat{\psi}_d|_{D_i} &= 0, \\ \psi_{3-i}|_{D_i} &= \psi_{3-i}, & \hat{\psi}_j|_{D_i} &= \hat{\psi}_j \quad \forall j = 1, \dots, d-1. \end{aligned} \quad (2.12)$$

If the left-hand side of (2.10) is not zero, the sum of the exponents is $d-1$. Thus, by symmetry, we can assume that $b_d=0$. By (2.11) and (2.12),

$$\int_{\overline{\mathcal{M}}_{0,2|d}} \psi_1^{a_1} \psi_2^{a_2} \hat{\psi}_1^{b_1} \dots \hat{\psi}_d^{b_d} = \int_{\overline{\mathcal{M}}_{0,2|d-1}} \psi_1^{a_1-1} \psi_2^{a_2} \hat{\psi}_1^{b_1} \dots \hat{\psi}_d^{b_d} + \int_{\overline{\mathcal{M}}_{0,2|d-1}} \psi_1^{a_1} \psi_2^{a_2-1} \hat{\psi}_1^{b_1} \dots \hat{\psi}_d^{b_d}.$$

This implies (2.10) by induction on d (if $d=1$, $\overline{\mathcal{M}}_{0,2|d}$ is a single point). \square

If $J, J' \subset \{1, \dots, d\}$, $|J|, |J'| \geq 2$, and $|J \cap J'| = 0, 1$, the “diagonals” Δ_J and $\Delta_{J'}$ intersect transversally; in particular, $\Delta_J \cdot \Delta_{J'} = \Delta_{J \cup J'}$ if $|J \cap J'| = 1$. Corollary 2.5 below partly describes this intersection in other cases as well.

Corollary 2.5. *Let $d \in \mathbb{Z}^+$.*

(1) *For all $J, J' \subset \{1, \dots, d\}$ with $|J|, |J'| \geq 2$,*

$$J \cap J' \neq \emptyset \quad \implies \quad \Delta_J \cdot \Delta_{J'} = \delta_{|J \cap J'|, 1} \Delta_{J \cup J'} \in \tilde{H}(\overline{\mathcal{M}}_{0,2|d}).$$

(2) *For all $j=1, \dots, d$, $\hat{\psi}_j = 0 \in \tilde{H}(\overline{\mathcal{M}}_{0,2|d})$.*

Proof. We prove both statements by simultaneous induction on d . If $d=1$, $\overline{\mathcal{M}}_{0,2|d}$ is a single point, and there is nothing to prove. Suppose $d \geq 2$ and the two statements hold with d replaced by any $d' \leq d-1$. If $J \subset \{1, \dots, d\}$ with $|J| \geq 2$,

$$\Delta_J \approx \overline{\mathcal{M}}_{0,2|(d-|J|+1)}; \quad (2.13)$$

such an isomorphism is determined by a surjection

$$\eta: \{1, \dots, d\} \longrightarrow \{1, \dots, d-|J|+1\} \quad \text{s.t.} \quad \eta(j_1) = \eta(j_2) \quad \forall j_1, j_2 \in J.$$

Under the isomorphism (2.13), $\psi_i|_{\Delta_J} = \psi_i$ for $i=1, 2$,

$$\hat{\psi}_j|_{\Delta_J} = \hat{\psi}_{\eta(j)}, \quad \Delta_{J'}|_{\Delta_J} = \begin{cases} \Delta_{\eta(J')}, & \text{if } J' \cap J = \emptyset; \\ (-\hat{\psi}_{\eta(J)})^{|J'|-1}, & \text{if } J' \subset J; \\ (-\hat{\psi}_{\eta(J)}^{|J' \cap J|-1} \Delta_{\eta(J \cup J')}), & \text{otherwise.} \end{cases}$$

Thus, for every subset $J' \subset \{1, \dots, d\}$ such that $|J \cap J'| \geq 2$ and every class $\Omega \in H'(\overline{\mathcal{M}}_{0,2|d})$ there exists $\Omega_{J,J'} \in H'(\overline{\mathcal{M}}_{0,2|(d-|J|+1)})$ such that

$$\int_{\overline{\mathcal{M}}_{0,2|d}} \Delta_J \Delta_{J'} \Omega \equiv \int_{\Delta_J} (\Delta_{J'} \Omega)|_{\Delta_J} = \int_{\overline{\mathcal{M}}_{0,2|(d-|J|+1)}} \hat{\psi}_{\eta(J)} \Omega_{J,J'} = 0;$$

the last equality holds by the inductive assumption that (2) is satisfied with d replaced by $d-|J|+1$. This confirms the first claim for the given choice of d . Similarly, if $j=1, \dots, d$ and $\Omega \in H'(\overline{\mathcal{M}}_{0,2|d})$, there exists $\Omega_{J,j} \in H'(\overline{\mathcal{M}}_{0,2|(d-|J|+1)})$ such that

$$\int_{\overline{\mathcal{M}}_{0,2|d}} \hat{\psi}_j \Delta_J \eta \equiv \int_{\Delta_J} (\hat{\psi}_j \Omega)|_{\Delta_J} = \int_{\overline{\mathcal{M}}_{0,2|(d-|J|+1)}} \hat{\psi}_{\eta(j)} \Omega_{J,j} = 0; \quad (2.14)$$

the last equality holds by the inductive assumption that (2) is satisfied with d replaced by $d-|J|+1$. Finally,

$$\int_{\overline{\mathcal{M}}_{0,2|d}} \hat{\psi}_j \psi_1^{a_1} \psi_2^{a_2} \hat{\psi}_1^{b_1} \dots \hat{\psi}_d^{b_d} = 0 \quad \forall a_1, a_2, b_1, \dots, b_d \in \mathbb{Z}^{\geq 0}$$

by Lemma 2.4. Along with (2.14), this confirms the second claim for the given choice of d . \square

Our proof of Theorems 1 and 3 immediately leads to a closed formula for certain twisted equivariant Hurwitz numbers stated in Theorem 4 stated in Section 4. We conclude this section with a non-equivariant version of this formula.

Let $x \in H^2(\mathbb{P}^\infty)$ denote the hyperplane class. For any $d \in \mathbb{Z}^+$, let

$$\mathcal{S}^*(x) \equiv \pi_{\mathbb{P}^\infty}^* \mathcal{O}_{\mathbb{P}^\infty}(1) \otimes \pi_{\mathcal{U}}^* \mathcal{S}^* \longrightarrow \mathbb{P}^\infty \times \mathcal{U} \longrightarrow \mathbb{P}^\infty \times \overline{\mathcal{M}}_{0,2|d},$$

where $\pi_{\mathbb{P}^\infty}, \pi_{\mathcal{U}}: \mathbb{P}^\infty \times \mathcal{U} \longrightarrow \mathbb{P}^\infty, \mathcal{U}$ are the two projections. In particular,

$$e(\mathcal{S}^*(x)) = x \times 1 + 1 \times e(\mathcal{S}^*) \in H^*(\mathbb{P}^\infty \times \mathcal{U}) = \mathbb{Q}[x] \otimes H^*(\mathcal{U}).$$

Similarly to (1.2), let

$$\mathcal{V}'_{\mathbf{a};d}(x) = \bigoplus_{a_k > 0} R^0 \pi_* (\mathcal{S}^*(x)^{a_k} (-\sigma_1)) \oplus \bigoplus_{a_k < 0} R^1 \pi_* (\mathcal{S}^*(x)^{a_k} (-\sigma_1)) \longrightarrow \overline{\mathcal{M}}_{0,2|d}, \quad (2.15)$$

where $\pi: \mathcal{U} \longrightarrow \overline{\mathcal{M}}_{0,2|d}$ is the projection as before; this sheaf is locally free.⁵ We define power series $L_{\mathbf{a}}, \xi_{\mathbf{a}} \in \mathbb{Q}[x][[q]]$ by

$$\begin{aligned} L_{\mathbf{a}} &\in x + q\mathbb{Q}[x][[q]], & L_{\mathbf{a}}(x, q) - q\mathbf{a}^{\mathbf{a}} L_{\mathbf{a}}(x, q)^{|\mathbf{a}|} &= x, \\ \xi_{\mathbf{a}} &\in q\mathbb{Q}[x][[q]], & x + q \frac{d}{dq} \xi_{\mathbf{a}}(x, q) &= L_{\mathbf{a}}(x, q). \end{aligned}$$

Theorem 2. *If $l \in \mathbb{Z}^{\geq 0}$ and $\mathbf{a} \in (\mathbb{Z}^*)^l$, then*

$$1 + (\hbar_1 + \hbar_2) \sum_{d=1}^{\infty} \frac{q^d}{d!} \int_{\overline{\mathcal{M}}_{0,2|d}} \frac{e(\mathcal{V}'_{\mathbf{a};d}(x))}{(\hbar_1 - \psi_1)(\hbar_2 - \psi_2)} = e^{\frac{\xi_{\mathbf{a}}(x, q)}{\hbar_1} + \frac{\xi_{\mathbf{a}}(x, q)}{\hbar_2}} \in \mathbb{Q}[x][[\hbar_1^{-1}, \hbar_2^{-1}, q]].$$

Proof. This is obtained from Theorem 4 by setting $n=1$, $i=1$, and $x=\alpha_i$. \square

⁵Note that the analogous push-down bundle (1.2) over $\overline{\mathcal{Q}}_{0,2}(\mathbb{P}^{n-1}, d)$ is denoted by $\mathcal{V}'_{n;\mathbf{a}}(d)$ to emphasize its dependence on n and highlight the distinction from the bundle $\mathcal{V}'_{\mathbf{a};d}(x)$ over $\overline{\mathcal{M}}_{0,2|d}$.

In the case $l=0$, the left-hand side of the expression in Theorem 2 reduces to

$$\begin{aligned} 1 + \sum_{a_1, a_2 \geq 0} (\hbar_1^{-a_1} \hbar_1^{-(a_2+1)} + \hbar_1^{-(a_1+1)} \hbar_1^{-a_2}) \frac{q^{a_1+a_2+1}}{(a_1+a_2+1)!} \int_{\overline{\mathcal{M}}_{0,2|a_1+a_2+1}} \psi_1^{a_1} \psi_2^{a_2} \\ = 1 + \sum_{a_1, a_2 \geq 0} (\hbar_1^{-a_1} \hbar_1^{-(a_2+1)} + \hbar_1^{-(a_1+1)} \hbar_1^{-a_2}) \frac{q^{a_1+a_2+1}}{(a_1+a_2+1)!} \binom{a_1+a_2}{a_1} = e^{\frac{q}{\hbar_1} + \frac{q}{\hbar_2}}; \end{aligned}$$

the first equality above holds by Lemma 2.4. Since $\xi_{\mathbf{a}}(x, q) = q$ in this case, this agrees with Theorem 2.

3 Equivariant cohomology

In this section, we review the notion of equivariant cohomology and set up related notation that will be used throughout the rest of the paper. For the most part, our notation agrees with [9, Chapters 29,30]; the main difference is that we work with \mathbb{P}^{n-1} instead of \mathbb{P}^n .

For any $n \in \mathbb{Z}^+$, let

$$[n] = \{1, \dots, n\}.$$

We denote by \mathbb{T} the n -torus $(\mathbb{C}^*)^n$. It acts freely on $E\mathbb{T} = (\mathbb{C}^\infty)^n - 0$:

$$(t_1, \dots, t_n) \cdot (z_1, \dots, z_n) = (t_1 z_1, \dots, t_n z_n).$$

Thus, the classifying space for \mathbb{T} and its group cohomology are given by

$$B\mathbb{T} \equiv E\mathbb{T}/\mathbb{T} = (\mathbb{P}^\infty)^n \quad \text{and} \quad H_{\mathbb{T}}^* \equiv H^*(B\mathbb{T}; \mathbb{Q}) = \mathbb{Q}[\alpha_1, \dots, \alpha_n],$$

where $\alpha_i = \pi_i^* c_1(\gamma^*)$ if

$$\pi_i: (\mathbb{P}^\infty)^n \longrightarrow \mathbb{P}^\infty \quad \text{and} \quad \gamma \longrightarrow \mathbb{P}^\infty$$

are the projection onto the i -th component and the tautological line bundle, respectively. Let

$$\mathcal{H}_{\mathbb{T}}^* = \mathbb{Q}_\alpha \equiv \mathbb{Q}(\alpha_1, \dots, \alpha_n)$$

the field of fractions of $H_{\mathbb{T}}^*$.

A representation ρ of \mathbb{T} , i.e. a linear action of \mathbb{T} on \mathbb{C}^k , induces a vector bundle over $B\mathbb{T}$:

$$V_\rho \equiv E\mathbb{T} \times_{\mathbb{T}} \mathbb{C}^k.$$

If ρ is one-dimensional, we will call

$$c_1(V_\rho^*) = -c_1(V_\rho) \in H_{\mathbb{T}}^* \subset \mathbb{Q}_\alpha$$

the **weight** of ρ . For example, α_i is the weight of the representation

$$\pi_i: \mathbb{T} \longrightarrow \mathbb{C}^*, \quad (t_1, \dots, t_n) \cdot z = t_i z. \quad (3.1)$$

More generally, if a representation ρ of \mathbb{T} on \mathbb{C}^k splits into one-dimensional representations with weights β_1, \dots, β_k , we will call β_1, \dots, β_k the **weights** of ρ . In such a case,

$$e(V_\rho^*) = \beta_1 \cdot \dots \cdot \beta_k. \quad (3.2)$$

We will call the representation ρ of \mathbb{T} on \mathbb{C}^n with weights $\alpha_1, \dots, \alpha_n$ the **standard representation** of \mathbb{T} .

If \mathbb{T} acts on a topological space M , let

$$H_{\mathbb{T}}^*(M) \equiv H^*(BM; \mathbb{Q}), \quad \text{where} \quad BM = E\mathbb{T} \times_{\mathbb{T}} M,$$

denote the corresponding **equivariant cohomology** of M . The projection map $BM \rightarrow B\mathbb{T}$ induces an action of $H_{\mathbb{T}}^*$ on $H_{\mathbb{T}}^*(M)$. Let

$$\mathcal{H}_{\mathbb{T}}^*(M) = H_{\mathbb{T}}^*(M) \otimes_{H_{\mathbb{T}}^*} \mathcal{H}_{\mathbb{T}}^*.$$

If the \mathbb{T} -action on M lifts to an action on a (complex) vector bundle $V \rightarrow M$, then

$$BV \equiv E\mathbb{T} \times_{\mathbb{T}} V$$

is a vector bundle over BM . Let

$$\mathbf{e}(V) \equiv e(BV) \in H_{\mathbb{T}}^*(M) \subset \mathcal{H}_{\mathbb{T}}^*(M)$$

denote the **equivariant euler class** of V .

Throughout the paper we work with the standard action of \mathbb{T} on \mathbb{P}^{n-1} , i.e. the action induced by the standard action ρ of \mathbb{T} on \mathbb{C}^n :

$$(t_1, \dots, t_n) \cdot [z_1, \dots, z_n] = [t_1 z_1, \dots, t_n z_n].$$

Since $B\mathbb{P}^{n-1} = \mathbb{P}V_\rho$,

$$H_{\mathbb{T}}^*(\mathbb{P}^{n-1}) \equiv H^*(\mathbb{P}V_\rho; \mathbb{Q}) = \mathbb{Q}[\mathbf{x}, \alpha_1, \dots, \alpha_n] / (\mathbf{x}^n + c_1(V_\rho)\mathbf{x}^{n-1} + \dots + c_n(V_\rho)),$$

where $\mathbf{x} = c_1(\tilde{\gamma}^*)$ and $\tilde{\gamma} \rightarrow \mathbb{P}V_\rho$ is the tautological line bundle. Since

$$c(V_\rho) = (1 - \alpha_1) \dots (1 - \alpha_n),$$

it follows that

$$\begin{aligned} H_{\mathbb{T}}^*(\mathbb{P}^{n-1}) &= \mathbb{Q}[\mathbf{x}, \alpha_1, \dots, \alpha_n] / (\mathbf{x} - \alpha_1) \dots (\mathbf{x} - \alpha_n), \\ \mathcal{H}_{\mathbb{T}}^*(\mathbb{P}^{n-1}) &= \mathbb{Q}_\alpha[\mathbf{x}] / (\mathbf{x} - \alpha_1) \dots (\mathbf{x} - \alpha_n). \end{aligned} \quad (3.3)$$

The standard action of \mathbb{T} on \mathbb{P}^{n-1} has n fixed points:

$$P_1 = [1, 0, \dots, 0], \quad P_2 = [0, 1, 0, \dots, 0], \quad \dots \quad P_n = [0, \dots, 0, 1].$$

For each $i = 1, 2, \dots, n$, let

$$\phi_i = \prod_{k \neq i} (\mathbf{x} - \alpha_k) \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1}). \quad (3.4)$$

By equation (3.9) below, ϕ_i is the equivariant Poincare dual of P_i . We also note that $\tilde{\gamma}|_{BP_i} = V_{\pi_i}$, where π_i is as in (3.1). Thus, the restriction map on the equivariant cohomology induced by the inclusion $P_i \rightarrow \mathbb{P}^{n-1}$ is given by

$$H_{\mathbb{T}}^*(\mathbb{P}^{n-1}) = \mathbb{Q}[\mathbf{x}, \alpha_1, \dots, \alpha_n] / \prod_{k=1}^{k=n} (\mathbf{x} - \alpha_k) \longrightarrow H_{\mathbb{T}}^*(P_i) = \mathbb{Q}[\alpha_1, \dots, \alpha_n], \quad \mathbf{x} \longrightarrow \alpha_i.$$

In particular, if $\mathcal{F} \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})$, then

$$\mathcal{F} = 0 \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1}) \iff \mathcal{F}(\mathbf{x} = \alpha_i) \equiv \mathcal{F}|_{P_i} = 0 \in \mathbb{Q}[\alpha_1, \dots, \alpha_n] \subset \mathbb{Q}_{\alpha} \quad \forall i \in [n]. \quad (3.5)$$

The tautological line bundle $\gamma_{n-1} \rightarrow \mathbb{P}^{n-1}$ is a subbundle of $\mathbb{P}^{n-1} \times \mathbb{C}^n$ preserved by the diagonal action of \mathbb{T} . Thus, the action of \mathbb{T} on \mathbb{P}^{n-1} naturally lifts to an action on γ_{n-1} and

$$\mathbf{e}(\gamma_{n-1}^*)|_{P_i} = \alpha_i \quad \forall i = 1, 2, \dots, n. \quad (3.6)$$

The \mathbb{T} -action on \mathbb{P}^{n-1} also has a natural lift to the vector bundle $T\mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n-1}$ so that there is a short exact sequence

$$0 \longrightarrow \gamma_{n-1}^* \otimes \gamma_{n-1} \longrightarrow \gamma_{n-1}^* \otimes (\mathbb{P}^{n-1} \times \mathbb{C}^n) \longrightarrow T\mathbb{P}^{n-1} \longrightarrow 0$$

of \mathbb{T} -equivariant vector bundles on \mathbb{P}^{n-1} . By (3.2), (3.6), and (3.4),

$$\mathbf{e}(T\mathbb{P}^{n-1})|_{P_i} = \prod_{k \neq i} (\alpha_i - \alpha_k) = \phi_i|_{P_i} \quad \forall i = 1, 2, \dots, n. \quad (3.7)$$

If \mathbb{T} acts smoothly on a smooth compact oriented manifold M , there is a well-defined integration-along-the-fiber homomorphism

$$\int_M : H_{\mathbb{T}}^*(M) \longrightarrow H_{\mathbb{T}}^*$$

for the fiber bundle $BM \rightarrow B\mathbb{T}$. The classical localization theorem of [1] relates it to integration along the fixed locus of the \mathbb{T} -action. The latter is a union of smooth compact orientable manifolds F ; \mathbb{T} acts on the normal bundle $\mathcal{N}F$ of each F . Once an orientation of F is chosen, there is a well-defined integration-along-the-fiber homomorphism

$$\int_F : H_{\mathbb{T}}^*(F) \longrightarrow H_{\mathbb{T}}^*.$$

The localization theorem states that

$$\int_M \eta = \sum_F \int_F \frac{\eta|_F}{\mathbf{e}(\mathcal{N}F)} \in \mathbb{Q}_{\alpha} \quad \forall \eta \in H_{\mathbb{T}}^*(M), \quad (3.8)$$

where the sum is taken over all components F of the fixed locus of \mathbb{T} . Part of the statement of (3.8) is that $\mathbf{e}(\mathcal{N}F)$ is invertible in $\mathcal{H}_{\mathbb{T}}^*(F)$. In the case of the standard action of \mathbb{T} on \mathbb{P}^{n-1} , (3.8) implies that

$$\eta|_{P_i} = \int_{\mathbb{P}^{n-1}} \eta \phi_i \in \mathbb{Q}_{\alpha} \quad \forall \eta \in \mathcal{H}_{\mathbb{T}}^*(\mathbb{P}^{n-1}), \quad i = 1, 2, \dots, n; \quad (3.9)$$

see also (3.7).

Finally, if $f: M \rightarrow M'$ is a \mathbb{T} -equivariant map between two compact oriented manifolds, there is a well-defined pushforward homomorphism

$$f_*: H_{\mathbb{T}}^*(M) \rightarrow H_{\mathbb{T}}^*(M').$$

It is characterized by the property that

$$\int_{M'} (f_*\eta) \eta' = \int_M \eta (f^*\eta') \quad \forall \eta \in H_{\mathbb{T}}^*(M), \eta' \in H_{\mathbb{T}}^*(M'). \quad (3.10)$$

The homomorphism \int_M of the previous paragraph corresponds to M' being a point. It is immediate from (3.10) that

$$f_*(\eta (f^*\eta')) = (f_*\eta) \eta' \quad \forall \eta \in H_{\mathbb{T}}^*(M), \eta' \in H_{\mathbb{T}}^*(M'). \quad (3.11)$$

4 Equivariant mirror theorem

In this section, we state an equivariant version of Theorems 1, Theorem 3, which immediately implies Theorems 1. It is proved in the rest of this paper, as outlined in Section 1 after the statement of Theorem 1. We then formulate an equivariant version of Theorem 2, Theorem 4, providing a closed formula for equivariant Hurwitz numbers. This theorem immediately implies Theorem 2 and is obtained in Section 8 by combining Proposition 8.3 in this paper with some results from [18].

The standard \mathbb{T} -representation on \mathbb{C}^n (as well as any other representation) induces a \mathbb{T} -action on the trivial rank n sheaf over any quasi-stable curve $(\mathcal{C}, y_1, \dots, y_m)$,

$$\mathbb{T} \cdot \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{C}} \rightarrow \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{C}}, \quad (t_1, \dots, t_n) \cdot (f_1, \dots, f_n) = (t_1 f_1, \dots, t_n f_n),$$

and thus on the subsheaves of this sheaf. This action preserves the rank and degree of the sheaf and the torsion and the stability properties of Section 2 and thus induces a \mathbb{T} -action on the moduli space $\overline{\mathcal{Q}}_{g,m}(\mathbb{G}(r, n), d)$, with respect to which the evaluation maps

$$\text{ev}_i: \overline{\mathcal{Q}}_{g,m}(\mathbb{G}(r, n), d) \rightarrow \mathbb{G}(r, n), \quad i = 1, 2, \dots, m,$$

are \mathbb{T} -equivariant. This action lifts to a \mathbb{T} -action on the universal subsheaf $\mathcal{S} \rightarrow \mathcal{U}$ and thus to \mathbb{T} -actions on the locally free sheaves

$$\pi_*(\sigma_i^2) \rightarrow \overline{\mathcal{Q}}_{g,m}(\mathbb{G}(r, n), d), \quad \mathcal{V}'_{n;\mathbf{a}}^{(d)} \rightarrow \overline{\mathcal{Q}}_{0,2}(\mathbb{P}^{n-1}, d).$$

This gives rise to \mathbb{T} -equivariant cohomology classes,

$$\psi_i \in H_{\mathbb{T}}^*(\overline{\mathcal{Q}}_{g,m}(\mathbb{P}^{n-1}, d)), \quad \mathbf{e}(\mathcal{V}'_{n;\mathbf{a}}^{(d)}) \in H_{\mathbb{T}}^*(\overline{\mathcal{Q}}_{0,2}(\mathbb{P}^{n-1}, d)).$$

The stable quotients analogue of the equivariant version of Givental's J -function is given by

$$\mathcal{Z}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q) \equiv 1 + \sum_{d=1}^{\infty} q^d \text{ev}_{1*} \left[\frac{\mathbf{e}(\mathcal{V}'_{n;\mathbf{a}}^{(d)})}{\hbar - \psi_1} \right] \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[[\hbar^{-1}, q]], \quad (4.1)$$

where $\text{ev}_1 : \overline{Q}_{0,2}(\mathbb{P}^{n-1}, d) \longrightarrow \mathbb{P}^{n-1}$ is as before. The equivariant analogue of the power series (1.5) is given by

$$\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q) \equiv \sum_{d=0}^{\infty} q^d \frac{\prod_{a_k > 0} \prod_{r=1}^{a_k d} (a_k \mathbf{x} + r \hbar) \prod_{a_k < 0} \prod_{r=0}^{-a_k d - 1} (a_k \mathbf{x} - r \hbar)}{\prod_{r=1}^d \prod_{k=1}^n (\mathbf{x} - \alpha_k + r \hbar)} \in \mathbb{Q}[\alpha_1, \dots, \alpha_n, \mathbf{x}][[\hbar^{-1}, q]]. \quad (4.2)$$

Theorem 3. *If $l \in \mathbb{Z}^{\geq 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$ are such that $|\mathbf{a}| \leq n$, then the equivariant stable quotients analogue of Givental's J -function satisfies*

$$\mathcal{Z}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q) = \frac{\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q)}{I_{n;\mathbf{a}}(q)} \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[[\hbar^{-1}, q]]. \quad (4.3)$$

Restricting to a fiber of the projection

$$B\mathbb{P}^{n-1} \equiv E\mathbb{T} \times_{\mathbb{T}} \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^{n-1},$$

we send \mathbf{x} to x and α_i to 0; this gives Theorem 1. The relation of Theorem 3 to its Gromov-Witten analogue is the same as the relation of Theorem 1 to its Gromov-Witten analogue; see the paragraph following the statement of Theorem 1 in Section 1. In particular, the twisted equivariant stable quotients invariants of \mathbb{P}^{n-1} determined by a tuple \mathbf{a} are the same as the corresponding Gromov-Witten invariants if $|\mathbf{a}| - \ell^-(\mathbf{a}) \leq n - 2$, but not if $|\mathbf{a}| - \ell^-(\mathbf{a}) = n - 1, n$.

We prove Theorem 3 through a two-pronged approach. We show that the power series $\mathcal{Y}_{n;\mathbf{a}}$ and $\mathcal{Z}_{n;\mathbf{a}}$ are \mathfrak{C} -recursive in the sense of Definition 5.1 with the collection \mathfrak{C} given by (5.6) and satisfy the self-polynomiality condition of Definition 5.2; see Lemma 5.4 and Propositions 6.1 and 7.1. Proposition 5.3 then implies that these power series are determined by their mod \hbar^{-2} -parts, i.e. the coefficients of \hbar^0 and \hbar^{-1} in this case. It is straightforward to determine the mod \hbar^{-2} -part of $\mathcal{Y}_{n;\mathbf{a}}$ in all cases ($\mathcal{Y}_{n;\mathbf{a}}$ is given by an explicit algebraic expression) and the mod \hbar^{-2} -part of $\mathcal{Z}_{n;\mathbf{a}}$ if $|\mathbf{a}| \leq n - 2$, thus establishing Theorem 3 whenever $|\mathbf{a}| \leq n - 2$; see Corollary 8.1.

In order to establish Theorem 3 in all cases, we show that the secondary coefficients $\mathcal{Y}_i^r(d)$ and $\mathcal{Z}_i^r(d)$ (instead of $\mathcal{F}_i^r(d)$) in the recursions (5.4) for $\mathcal{Y}_{n;\mathbf{a}}$ and $\mathcal{Z}_{n;\mathbf{a}}$ are the same. By induction on d , this implies that the coefficients of q^d on the two sides of (4.3) are the same because this is the case for $d = 0$ (when both coefficients are 1). As part of the proof of \mathfrak{C} -recursivity for $\mathcal{Y}_{n;\mathbf{a}}$, we show that $\mathcal{Y}_i^r(d)$ is determined by the expansion of $\mathcal{Y}_{n;\mathbf{a}}(\alpha_i, \hbar, q)$ around $\hbar = 0$; see Lemma 5.4. As part of the proof of \mathfrak{C} -recursivity for $\mathcal{Z}_{n;\mathbf{a}}$, we show that $\mathcal{Z}_i^r(d)$ is also determined by the expansion of $\mathcal{Z}_{n;\mathbf{a}}(\alpha_i, \hbar, q)$ around $\hbar = 0$; see Proposition 6.1. In contrast to $\mathcal{Y}_i^r(d)$, $\mathcal{Z}_i^r(d)$ is determined by lower-degree coefficients of $\mathcal{Z}_{n;\mathbf{a}}$ or equivalently by $\mathcal{Z}_j^s(d')$ with $d' < d$; this relation thus completely determines $\mathcal{Z}_{n;\mathbf{a}}$ (assuming \mathfrak{C} -recursivity). It follows that (4.3) holds if and only if the coefficients $\mathcal{Y}_i^r(d)$ for $\mathcal{Y}_{n;\mathbf{a}}$ satisfy the same relation; see Lemma 8.2. The coefficients in this relation involve twisted Hurwitz numbers over the moduli spaces $\overline{\mathcal{M}}_{0,2|d}$. These are not easy to compute, but using Corollary 2.5 they can be described qualitatively in way independent of n . This implies that the validity of the desired recursion for the secondary coefficients $\mathcal{Y}_i^r(d)$ for $\mathcal{Y}_{n;\mathbf{a}}$ is *independent* of n . Since this recursion is equivalent to (4.3) whenever $|\mathbf{a}| \leq n$ and (4.3) holds whenever $|\mathbf{a}| \leq n - 2$ (by Corollary 8.1), it follows that the recursion holds in all cases (see Proposition 8.3) and (4.3) holds

whenever $|\mathbf{a}| \leq n$, as claimed.

As stated in Section 1, Theorem 3 extends to products of projective spaces and concavex sheaves (1.7). The relevant torus action is then the product of the actions on the components described in Section 3. If its weights are denoted by $\alpha_{i;j}$, with $i=1, \dots, p$ and $j=1, \dots, n_i$, then

$$\mathcal{Y}_{n_1, \dots, n_p; \mathbf{a}}(\mathbf{x}_1, \dots, \mathbf{x}_p, \hbar, q_1, \dots, q_p) \in \mathbb{Q}[\alpha_{1;1}, \dots, \alpha_{p;n_p}, \mathbf{x}_1, \dots, \mathbf{x}_p][[\hbar^{-1}, q_1, \dots, q_p]], \quad (4.4)$$

$$\mathcal{Z}_{n_1, \dots, n_p; \mathbf{a}}(\mathbf{x}_1, \dots, \mathbf{x}_p, \hbar, q_1, \dots, q_p) \in H_{\mathbb{T}}^*(\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_p-1})[[\hbar^{-1}, q_1, \dots, q_p]], \quad (4.5)$$

and $\mathbf{x}_1, \dots, \mathbf{x}_p \in H^*(\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_p-1})$ correspond to the pullbacks of the equivariant hyperplane classes by the projection maps. The coefficient of $q_1^{d_1} \dots q_p^{d_p}$ in (4.5) is defined by the same pushforward as in (4.1), with the degree d of the stable maps replaced by (d_1, \dots, d_p) . The coefficient of $q_1^{d_1} \dots q_p^{d_p}$ in (4.4) is obtained from the coefficients in (4.2) by replacing $a_k d$ and $a_k \mathbf{x}$ by $a_{k;1} d_1 + \dots + a_{k;p} d_p$ and $a_{k;1} \mathbf{x}_1 + \dots + a_{k;p} \mathbf{x}_p$ in the numerator and taking the product of the denominators with $(n, \mathbf{x}, d) = (n_s, \mathbf{x}_s, d_s)$ for each $s=1, \dots, p$; in the s -th factor, α_k is also replaced by $\alpha_{s;k}$. Our proof of Theorem 3 extends directly to this situation.

We conclude this section with an equivariant version of Theorem 2. For any $d \in \mathbb{Z}^+$ and $\beta \in H_{\mathbb{T}}^2$, denote by

$$\mathcal{S}^*(\beta) \longrightarrow \mathcal{U} \longrightarrow \overline{\mathcal{M}}_{0,2|d} \quad (4.6)$$

the universal sheaf with the \mathbb{T} -action so that

$$\mathbf{e}(\mathcal{S}^*(\beta)) = \beta \times 1 + 1 \times e(\mathcal{S}^*) \in H_{\mathbb{T}}^*(\mathcal{U}) = H_{\mathbb{T}}^* \otimes H^*(\mathcal{U}).$$

Similarly to (1.2), let

$$\mathcal{V}'_{\mathbf{a};d}(\beta) = \bigoplus_{a_k > 0} R^0 \pi_* (\mathcal{S}^*(\beta)^{a_k} (-\sigma_1)) \oplus \bigoplus_{a_k < 0} R^1 \pi_* (\mathcal{S}^*(\beta)^{a_k} (-\sigma_1)) \longrightarrow \overline{\mathcal{M}}_{0,2|d}, \quad (4.7)$$

where $\pi: \mathcal{U} \longrightarrow \overline{\mathcal{M}}_{0,2|d}$ is the projection as before; this sheaf is locally free. We define power series $L_{n;\mathbf{a}}, \xi_{n;\mathbf{a}} \in \mathbb{Q}[\mathbf{x}][[q]]$ by

$$\begin{aligned} L_{n;\mathbf{a}} &\in \mathbf{x} + q \mathbb{Q}_{\alpha}[\mathbf{x}][[q]], & \prod_{k=1}^n (L_{n;\mathbf{a}}(\mathbf{x}, q) - \alpha_k) - q \mathbf{a}^{\mathbf{a}} L_{n;\mathbf{a}}(\mathbf{x}, q)^{|\mathbf{a}|} &= \prod_{k=1}^n (\mathbf{x} - \alpha_k), \\ \xi_{n;\mathbf{a}} &\in q \mathbb{Q}_{\alpha}[\mathbf{x}][[q]], & \mathbf{x} + q \frac{d}{dq} \xi_{n;\mathbf{a}}(\mathbf{x}, q) &= L_{n;\mathbf{a}}(\mathbf{x}, q). \end{aligned}$$

Theorem 4. *If $l \in \mathbb{Z}^{\geq 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$, then*

$$\begin{aligned} 1 + (\hbar_1 + \hbar_2) \sum_{d=1}^{\infty} \frac{q^d}{d!} \int_{\overline{\mathcal{M}}_{0,2|d}} \frac{\mathbf{e}(\mathcal{V}'_{\mathbf{a};d}(\alpha_i))}{\prod_{k \neq i} \mathbf{e}(\mathcal{V}'_{1;d}(\alpha_i - \alpha_k)) (\hbar_1 - \psi_1) (\hbar_2 - \psi_2)} \\ = e^{\frac{\xi_{n;\mathbf{a}}(\alpha_i, q)}{\hbar_1} + \frac{\xi_{n;\mathbf{a}}(\alpha_i, q)}{\hbar_2}} \in \mathbb{Q}_{\alpha}[[\hbar_1^{-1}, \hbar_2^{-1}, q]] \end{aligned}$$

for every $i=1, \dots, n$.

5 Algebraic observations

In this section we describe a number of properties of power series such as $\mathcal{Y}_{n;\mathbf{a}}$ and $\mathcal{Z}_{n;\mathbf{a}}$ that determine them uniquely. We also show that $\mathcal{Y}_{n;\mathbf{a}}$ indeed satisfies these properties.

If R is a ring, denote by

$$R[[\hbar]] \equiv R[[\hbar^{-1}]] + R[\hbar]$$

the R -algebra of Laurent series in \hbar^{-1} (with finite principal part). If $f \in R[[q]]$, and $d \in \mathbb{Z}^{\geq 0}$, let $[[f]]_{q;d} \in R$ denote the coefficient of q^d in f . If

$$\mathcal{F}(\hbar, q) = \sum_{d=0}^{\infty} \left(\sum_{r=-N_d}^{\infty} \mathcal{F}_d^{(r)} \hbar^{-r} \right) q^d \in R[[\hbar]][[q]]$$

for some $\mathcal{F}_d^{(r)} \in R$, we define

$$\mathcal{F}(\hbar, q) \cong \sum_{d=0}^{\infty} \left(\sum_{r=-N_d}^1 \mathcal{F}_d^{(r)} \hbar^{-r} \right) q^d \pmod{\hbar^{-p}},$$

i.e. we drop \hbar^{-p} and higher powers of \hbar^{-1} , instead of higher powers of \hbar . If R is a field, let

$$R(\hbar) \hookrightarrow R[[\hbar]]$$

be the embedding given by taking the Laurent series of rational functions at $\hbar^{-1}=0$.

If $f=f(z)$ is a rational function in z and possibly some other variables, for any $z_0 \in \mathbb{P}^1 \supset \mathbb{C}$ let

$$\Re_{z=z_0} f(z) \equiv \frac{1}{2\pi i} \oint f(z) dz, \quad (5.1)$$

where the integral is taken over a positively oriented loop around $z = z_0$ with no other singular points of $f dz$, denote the residue of the 1-form $f dz$. If $z_1, \dots, z_k \in \mathbb{P}^1$ is any collection of points, let

$$\Re_{z=z_1, \dots, z_k} f(z) \equiv \sum_{i=1}^k \Re_{z=z_i} f(z). \quad (5.2)$$

By the Residue Theorem on S^2 ,

$$\sum_{\mathbf{x}_0 \in S^2} \Re_{\mathbf{x}=\mathbf{x}_0} \{f(\mathbf{x})\} = 0$$

for every rational function $f=f(\mathbf{x})$ on $S^2 \supset \mathbb{C}$. If f is regular at $z=0$, let $[[f]]_{z;p}$ denote the coefficient of z^p in the power series expansion of f around $z=0$.

Definition 5.1. Let $C \equiv (C_i^j(d))_{d,i,j \in \mathbb{Z}^+}$ be any collection of elements of \mathbb{Q}_α . A power series $\mathcal{F} \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[[\hbar]][[q]]$ is C -recursive if the following holds: if $d^* \in \mathbb{Z}^{\geq 0}$ is such that

$$[[\mathcal{F}(\mathbf{x}=\alpha_i, \hbar, q)]]_{q;d^*-d} \in \mathbb{Q}_\alpha(\hbar) \subset \mathbb{Q}_\alpha[[\hbar]] \quad \forall d \in [d^*], i \in [n],$$

and $[[\mathcal{F}(\alpha_i, \hbar, q)]]_{q;d}$ is regular at $\hbar=(\alpha_i-\alpha_j)/d$ for all $d < d^*$ and $i \neq j$, then

$$[[\mathcal{F}(\alpha_i, \hbar, q)]]_{q;d^*} - \sum_{d=1}^{d^*} \sum_{j \neq i} \frac{C_i^j(d)}{\hbar - \frac{\alpha_j - \alpha_i}{d}} [[\mathcal{F}(\alpha_j, \hbar, q)]]_{q;d^*-d} \Big|_{\hbar = \frac{\alpha_j - \alpha_i}{d}} \in \mathbb{Q}_\alpha[\hbar, \hbar^{-1}] \subset \mathbb{Q}_\alpha[[\hbar]]. \quad (5.3)$$

Thus, if $\mathcal{F} \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[[\hbar]][[q]]$ is C -recursive, for any collection C , then

$$\mathcal{F}(\mathbf{x}=\alpha_i, \hbar, q) \in \mathbb{Q}_\alpha(\hbar)[[q]] \subset \mathbb{Q}_\alpha[[\hbar]][[q]] \quad \forall i \in [n],$$

as can be seen by induction on d , and

$$\mathcal{F}(\alpha_i, \hbar, q) = \sum_{d=0}^{\infty} \sum_{r=-N_d}^{N_d} \mathcal{F}_i^r(d) \hbar^r q^d + \sum_{d=1}^{\infty} \sum_{j \neq i} \frac{C_i^j(d) q^d}{\hbar - \frac{\alpha_j - \alpha_i}{d}} \mathcal{F}(\alpha_j, (\alpha_j - \alpha_i)/d, q) \quad \forall i \in [n], \quad (5.4)$$

for some $\mathcal{F}_i^r(d) \in \mathbb{Q}_\alpha$. The nominal issue with defining C -recursivity by (5.4), as is normally done, is that a priori the evaluation of $\mathcal{F}(\alpha_j, \hbar, q)$ at $\hbar = (\alpha_j - \alpha_i)/d$ need not be well-defined, since $\mathcal{F}(\alpha_j, \hbar, q)$ is a power series in q with coefficients in the Laurent series in \hbar^{-1} ; a priori they may not converge anywhere. However, taking the coefficient of each power of q in (5.4) shows by induction on the degree d that this evaluation does make sense; this is the substance of Definition 5.1.

Definition 5.2. For any $\mathcal{F} \equiv \mathcal{F}(\mathbf{x}, \hbar, q) \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[[\hbar]][[q]]$, let

$$\Phi_{\mathcal{F}}(\hbar, z, q) \equiv \sum_{i=1}^n \frac{\langle \mathbf{a} \rangle \alpha_i^{\ell(\mathbf{a})} e^{\alpha_i z}}{\prod_{k \neq i} (\alpha_i - \alpha_k)} \mathcal{F}(\alpha_i, \hbar, q e^{\hbar z}) \mathcal{F}(\alpha_i, -\hbar, q) \in \mathbb{Q}_\alpha[[\hbar]][[z, q]]. \quad (5.5)$$

A power series $\mathcal{F} \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[[\hbar]][[z, q]]$ satisfies the self-polynomiality condition if $\Phi_{\mathcal{F}} \in \mathbb{Q}_\alpha[\hbar][[z, q]]$.

Proposition 5.3 ([9, Lemma 30.3.2]). *Let $\mathcal{F}, \mathcal{F}' \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[[\hbar]][[q]]$. If \mathcal{F} and \mathcal{F}' are C -recursive, for some collection $C \equiv (C_i^j(d))_{d,i,j \in \mathbb{Z}^+}$ of elements of \mathbb{Q}_α , satisfy the self-polynomiality condition, and*

$$\mathcal{F}(\mathbf{x}=\alpha_i, \hbar, q), \mathcal{F}'(\mathbf{x}=\alpha_i, \hbar, q) \in \mathbb{Q}_\alpha^* + q \cdot \mathbb{Q}_\alpha[[\hbar]][[q]] \subset \mathbb{Q}_\alpha[[\hbar]][[q]] \quad \forall i \in [n],$$

then $\mathcal{F} \cong \mathcal{F}' \pmod{\hbar^{-2}}$ if and only if $\mathcal{F} = \mathcal{F}'$.

Let

$$\mathfrak{C}_i^j(d) \equiv \frac{\prod_{a_k > 0} \prod_{r=1}^{a_k d} \left(a_k \alpha_i + r \frac{\alpha_j - \alpha_i}{d} \right) \prod_{a_k < 0} \prod_{r=0}^{-a_k d - 1} \left(a_k \alpha_i - r \frac{\alpha_j - \alpha_i}{d} \right)}{d \prod_{r=1}^d \prod_{\substack{k=1 \\ (r,k) \neq (d,j)}}^n \left(\alpha_i - \alpha_k + r \frac{\alpha_j - \alpha_i}{d} \right)} \in \mathbb{Q}_\alpha. \quad (5.6)$$

Lemma 5.4. *If $l \in \mathbb{Z}^{\geq 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$ are such that $|\mathbf{a}| \leq n$, the power series $\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q)$ is \mathfrak{C} -recursive, with the auxiliary coefficients in the recursion (5.4) for $\mathcal{Y}_{n;\mathbf{a}}$ given by*

$$\sum_{d=0}^{\infty} \mathcal{Y}_i^r(d) q^d = \begin{cases} \Re_{\hbar=0} \{ \hbar^{-r-1} \mathcal{Y}_{n;\mathbf{a}}(\alpha_i, \hbar, q) \}, & \text{if } r < 0; \\ I_{n;\mathbf{a}}(q), & \text{if } r = 0; \\ 0, & \text{if } r > 0. \end{cases} \quad (5.7)$$

Furthermore, $\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q)$ satisfies the self-polynomiality condition.

Proof. This is well-known from the various proofs of mirror symmetry for Gromov-Witten invariants (e.g. [6, Section 11], [9, Chapter 30], [4, Section 4]); we include a proof for the sake of completeness.

We first view $\mathcal{Y}_{n;\mathbf{a}}$ as an element of $\mathbb{Q}_\alpha(\mathbf{x}, \hbar)[[q]]$. Since

$$\frac{\mathfrak{C}_i^j(d)q^d}{\hbar - \frac{\alpha_j - \alpha_i}{d}} \mathcal{Y}_{n;\mathbf{a}}(\alpha_j, (\alpha_j - \alpha_i)/d, q) = \mathfrak{R}_{z=\frac{\alpha_j - \alpha_i}{d}} \left\{ \frac{1}{\hbar - z} \mathcal{Y}_{n;\mathbf{a}}(\alpha_i, z, q) \right\},$$

by the Residue Theorem on S^2

$$\begin{aligned} \sum_{d=1}^{\infty} \sum_{j \neq i} \frac{\mathfrak{C}_i^j(d)q^d}{\hbar - \frac{\alpha_j - \alpha_i}{d}} \mathcal{Y}_{n;\mathbf{a}}(\alpha_j, (\alpha_j - \alpha_i)/d, q) &= - \mathfrak{R}_{z=\hbar, 0, \infty} \left\{ \frac{1}{\hbar - z} \mathcal{Y}_{n;\mathbf{a}}(\alpha_i, z, q) \right\} \\ &= \mathcal{Y}_{n;\mathbf{a}}(\alpha_i, \hbar, q) - \mathfrak{R}_{z=0, \infty} \left\{ \frac{1}{\hbar - z} \mathcal{Y}_{n;\mathbf{a}}(\alpha_i, z, q) \right\}. \end{aligned} \quad (5.8)$$

On the other hand,

$$\begin{aligned} \mathfrak{R}_{z=\infty} \left\{ \frac{1}{\hbar - z} \mathcal{Y}_{n;\mathbf{a}}(\alpha_i, z, q) \right\} &= I_{n;\mathbf{a}}(q), \\ \mathfrak{R}_{z=0} \left\{ \frac{1}{\hbar - z} \mathbb{I}_{\mathcal{Y}_{n;\mathbf{a}}(\alpha_i, z, q)} \right\} &= \left\| \frac{1}{\hbar - z} \frac{\prod_{a_k > 0} \prod_{r=1}^{a_k d} (a_k \alpha_i + rz) \prod_{a_k < 0} \prod_{r=0}^{-a_k d - 1} (a_k \alpha_i - rz)}{d! \prod_{r=1}^d \prod_{k \neq i} (\alpha_i - \alpha_k + rz)} \right\|_{z; d-1}; \end{aligned}$$

the last expression is a polynomial in \hbar^{-1} with coefficients in \mathbb{Q}_α of degree at most d . This establishes that $\mathcal{Y}_{n;\mathbf{a}}$ is \mathfrak{C} -recursive and (5.7) holds.

We now view $\mathcal{Y}_{n;\mathbf{a}}$ as an element of $\mathbb{Q}_\alpha[\mathbf{x}][[\hbar^{-1}, q]]$; in particular,

$$\frac{\langle \mathbf{a} \rangle \alpha_i^{\ell(\mathbf{a})} e^{\mathbf{x}z}}{\prod_{k=1}^n (\mathbf{x} - \alpha_k)} \mathcal{Y}_{n;\mathbf{a}}(\mathbf{x}, \hbar, qe^{\hbar z}) \mathcal{Y}_{n;\mathbf{a}}(-\mathbf{x}, \hbar, q) \in \mathbb{Q}_\alpha(\mathbf{x})[[\hbar^{-1}, z, q]]$$

viewed as a function of \mathbf{x} has residues only at $\mathbf{x} = \alpha_i$ with $i \in [n]$ and $\mathbf{x} = \infty$. By (4.2),

$$\frac{\langle \mathbf{a} \rangle \alpha_i^{\ell(\mathbf{a})} e^{\alpha_i z}}{\prod_{k \neq i} (\alpha_i - \alpha_k)} \mathcal{Y}_{n;\mathbf{a}}(\alpha_i, \hbar, qe^{\hbar z}) \mathcal{Y}_{n;\mathbf{a}}(\alpha_i, -\hbar, q) = \mathfrak{R}_{\mathbf{x}=\alpha_i} \left\{ \frac{\langle \mathbf{a} \rangle \mathbf{x}^{\ell(\mathbf{a})} e^{\mathbf{x}z}}{\prod_{k=1}^n (\mathbf{x} - \alpha_k)} \mathcal{Y}_{n;\mathbf{a}}(\mathbf{x}, \hbar, qe^{\hbar z}) \mathcal{Y}_{n;\mathbf{a}}(\mathbf{x}, -\hbar, q) \right\}.$$

Thus, by the Residue Theorem on S^2 ,

$$\Phi_{\mathcal{Y}_{n;\mathbf{a}}}(\hbar, z, q) = - \mathfrak{R}_{\mathbf{x}=0, \infty} \left\{ \frac{\langle \mathbf{a} \rangle \mathbf{x}^{\ell(\mathbf{a})} e^{\mathbf{x}z}}{\prod_{k=1}^n (\mathbf{x} - \alpha_k)} \mathcal{Y}_{n;\mathbf{a}}(\mathbf{x}, \hbar, qe^{\hbar z}) \mathcal{Y}_{n;\mathbf{a}}(\mathbf{x}, -\hbar, q) \right\} \equiv -\mathfrak{R}_0 - \mathfrak{R}_\infty.$$

Since the coefficients of positive powers of q in $\mathcal{Y}_{n;\mathbf{a}}$ are divisible by $\mathbf{x}^{\ell^-(\mathbf{a})}$,

$$\mathfrak{R}_0 = \langle \mathbf{a} \rangle \left[\left[\frac{e^{\mathbf{x}z}}{\prod_{k=1}^n (\mathbf{x} - \alpha_k)} \right]_{\mathbf{x}; -\ell(\mathbf{a})-1} \right] \in \mathbb{Q}_\alpha[z] \subset \mathbb{Q}_\alpha[\hbar][[z, q]].$$

On the other hand,

$$\begin{aligned} -\mathfrak{R}_\infty &= \langle \mathbf{a} \rangle \sum_{d_1, d_2=0}^{\infty} \sum_{p=0}^{\infty} \frac{z^{n-1-\ell(\mathbf{a})+p+(n-|\mathbf{a}|)(d_1+d_2)}}{(n-1-\ell(\mathbf{a})+p+(n-|\mathbf{a}|)(d_1+d_2))!} q^{d_1+d_2} e^{\hbar d_1 z} \\ &\times \left[\frac{\prod_{a_k > 0} \prod_{r=1}^{a_k d_1} (a_k + r \hbar w) \prod_{a_k < 0} \prod_{r=0}^{-a_k d_1 - 1} (a_k - r \hbar w) \cdot \prod_{a_k > 0} \prod_{r=1}^{a_k d_2} (a_k - r \hbar w) \prod_{a_k < 0} \prod_{r=0}^{-a_k d_2 - 1} (a_k + r \hbar w)}{\prod_{k=1}^n (1 - \alpha_k w) \cdot \prod_{r=1}^{d_1} \prod_{k=1}^n (1 - (\alpha_k - r \hbar) w) \prod_{r=1}^{d_2} \prod_{k=1}^n (1 - (\alpha_k + r \hbar) w)} \right]_{w; p}. \end{aligned}$$

The (d_1, d_2, p) -summand above is $q^{d_1+d_2}$ times an element of $\mathbb{Q}_\alpha[\hbar][[z]]$. \square

In the case of products of projective spaces and concavex sheaves (1.7), Definition 5.1 becomes inductive on the total degree $d_1 + \dots + d_p$ of $q_1^{d_1} \dots q_p^{d_p}$. The power series \mathcal{F} is evaluated at $(\mathbf{x}_1, \dots, \mathbf{x}_p) = (\alpha_{1;i_1}, \dots, \alpha_{p;i_p})$ for the purposes of the C -recursivity condition (5.3) and (5.4) and the primary structure coefficients are of the form

$$\mathfrak{C}_{i_1 \dots i_p}^j(s; d) \equiv \frac{\prod_{a_{k;1} > 0} \prod_{r=1}^{a_{k;s} d} \left(\sum_{t=1}^p a_{k;t} \alpha_{t;i_t} + r \frac{\alpha_{s;j} - \alpha_{s;i_s}}{d} \right) \prod_{a_{k;1} < 0} \prod_{r=0}^{-a_{k;s} d - 1} \left(\sum_{t=1}^p a_{k;t} \alpha_{t;i_t} - r \frac{\alpha_{s;j} - \alpha_{s;i_s}}{d} \right)}{d \prod_{r=1}^d \prod_{k=1}^{n_s} \left(\alpha_{s;i_s} - \alpha_{s;k} + r \frac{\alpha_{s;j} - \alpha_{s;i_s}}{d} \right)}_{(r,k) \neq (d,j)}$$

with $s \in [p]$ and $j \neq i_s$. The double sums in these equations are then replaced by triple sums over $s \in [p]$, $j \in [n_s] - i_s$, and $d \in \mathbb{Z}^+$, and with \mathcal{F} evaluated at

$$\mathbf{x}_t = \begin{cases} \alpha_{s;j}, & \text{if } t = s; \\ \alpha_{t;i_t}, & \text{if } t \neq s; \end{cases} \quad z = \frac{\alpha_{s;j} - \alpha_{s;i_s}}{d}.$$

The secondary coefficients $\mathcal{F}_i^r(d)$ in (5.4) now become $\mathcal{F}_{i_1 \dots i_p}^r(d_1, \dots, d_p)$, with $i_s \in [n_s]$ and $d_s \in \mathbb{Z}^{\geq 0}$. In the analogue of Definition 5.2, $\Phi_{\mathcal{F}}$ is a power series in z_1, \dots, z_p and q_1, \dots, q_p , the sum taken is over all elements (i_1, \dots, i_p) of $[n_1] \times \dots \times [n_p]$, the leading fraction is replaced by

$$\frac{\prod_{a_{k;1} > 0} \sum_{s=1}^p a_{k;s} \alpha_{s;i_s}}{\prod_{a_{k;1} < 0} \sum_{s=1}^p a_{k;s} \alpha_{s;i_s}} \cdot \frac{e^{\alpha_{1;i_1} z_1 + \dots + \alpha_{p;i_p} z_p}}{\prod_{s=1}^p \prod_{k \neq i_s} (\alpha_{s;i_s} - \alpha_{s;k})},$$

and the $qe^{\hbar z}$ -insertion in the first power series is replaced by the insertions $q_1 e^{\hbar z_1}, \dots, q_p e^{\hbar z_p}$. The conclusion of Lemma 5.4 holds with i , d , and q^d replaced by (i_1, \dots, i_p) , (d_1, \dots, d_p) , and $q_1^{d_1} \dots q_p^{d_p}$,

respectively. The proof is nearly identical, except the last claim involves p applications of the Residue Theorem on S^2 . Instead of the residue at $\mathbf{x} = 0$ of the coefficient of q^0 , there may be a residue at a value of \mathbf{x}_s dependent on the values of the other variables \mathbf{x}_t , but it again would not involve \hbar .

6 Recursivity for stable quotients

In this section, we use the classical localization theorem [1] to show that the equivariant stable quotients analogue of Givental's J -function, the power series $\mathcal{Z}_{n;\mathbf{a}}$ given by (4.1), is \mathfrak{C} -recursive with the collection $\mathfrak{C}_i^j(d)$ given by (5.6). We also describe the secondary terms $\mathcal{Z}_i^r(d)$ in the recursion (5.4) for $\mathcal{Z}_{n;\mathbf{a}}$, establishing the following statement.

Proposition 6.1. *If $l \in \mathbb{Z}^{\geq 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$, the power series $\mathcal{Z}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q)$ is \mathfrak{C} -recursive, with the auxiliary coefficients in the recursion (5.4) for $\mathcal{Z}_{n;\mathbf{a}}$ given by*

$$\mathcal{Z}_i^r(d) = 0 \quad \forall r \in \mathbb{Z}^+, \quad \mathcal{Z}_i^0(d) = \delta_{0d},$$

and for all $r \in \mathbb{Z}^-$

$$\sum_{d=1}^{\infty} \mathcal{Z}_i^r(d) q^d = \sum_{d=1}^{\infty} \frac{q^d}{d!} \sum_{b=0}^{d+r} \left(\left(\int_{\mathcal{M}_{0,2|d}} \frac{\mathbf{e}(\mathcal{V}'_{\mathbf{a};d}(\alpha_i)) \psi_1^{-r-1} \psi_2^b}{\prod_{k \neq i} \mathbf{e}(\mathcal{V}'_{1;d}(\alpha_i - \alpha_k))} \right) \Re \left\{ \frac{(-1)^b}{\hbar^{b+1}} \mathcal{Z}_{n;\mathbf{a}}(\alpha_i, \hbar, q) \right\} \right).$$

The proof involves a localization computation on $\overline{\mathcal{Q}}_{0,2}(\mathbb{P}^{n-1}, d)$. Thus, we need to describe the fixed loci of the \mathbb{T} -action on $\overline{\mathcal{Q}}_{0,2}(\mathbb{P}^{n-1}, d)$, their normal bundles, and the restrictions of the relevant cohomology classes to these fixed loci.

As in the case of stable maps described in [9, Section 27.3], the fixed loci of the \mathbb{T} -action on $\overline{\mathcal{Q}}_{0,m}(\mathbb{P}^{n-1}, d)$ are indexed by **decorated graphs** that have no loops. However, in the case $m = 2$, the relevant graphs consist of a single **strand** (possibly consisting of a single vertex) with the two marked points attached at the opposite ends of the strand. Such a graph can be described by an ordered set $(\text{Ver}, <)$ of vertices, where $<$ is a strict order on the finite set Ver . Given such a strand, denote by v_{\min} and v_{\max} its minimal and maximal elements and by Edg its set of **edges**, i.e. of pairs of consecutive elements. A **decorated strand** is a tuple

$$\Gamma = (\text{Ver}, <; \mu, \mathfrak{d}), \tag{6.1}$$

where $(\text{Ver}, <)$ is a strand as above and

$$\mu: \text{Ver} \longrightarrow [n] \quad \text{and} \quad \mathfrak{d}: \text{Ver} \sqcup \text{Edg} \longrightarrow \mathbb{Z}^{\geq 0}$$

are maps such that

$$\mu(v_1) \neq \mu(v_2) \quad \text{if } \{v_1, v_2\} \in \text{Edg}, \quad \mathfrak{d}(e) \neq 0 \quad \forall e \in \text{Edg}. \tag{6.2}$$

In Figure 1, the vertices of a decorated strand Γ are indicated by dots in the increasing order, with respect to $<$, from left to right. The values of the map (μ, \mathfrak{d}) on some of the vertices are indicated next to those vertices. Similarly, the values of the map \mathfrak{d} on some of the edges are indicated next

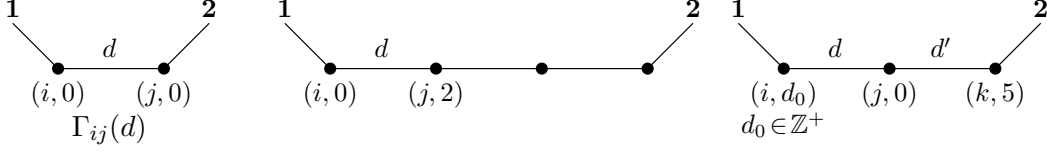


Figure 1: Two strands with $\mathfrak{d}(v_{\min})=0$ and a strand with $\mathfrak{d}(v_{\min})>0$

to them. By (6.2), no two consecutive vertices have the same first label and thus $j \neq i$. With Γ as in (6.1), let

$$|\Gamma| \equiv \sum_{v \in \text{Ver}} \mathfrak{d}(v) + \sum_{e \in \text{Edg}} \mathfrak{d}(e)$$

be the degree of Γ . If $e = \{v_1, v_2\} \in \text{Edg}$ is any edge in Γ , let Γ_e denote the single-edge graph with vertices v_1 and v_2 , which are ordered in the same way as in Γ and assigned values $(\mu(v_1), 0)$ and $(\mu(v_2), 0)$, and with the edge assigned the value $\mathfrak{d}(e)$ as in the original graph; see Figure 2.

As described in [13, Section 7.3], the fixed locus Q_Γ of $\overline{Q}_{0,2}(\mathbb{P}^{n-1}, |\Gamma|)$ corresponding to a decorated strand Γ consists of the stable quotients

$$(\mathcal{C}, y_1, y_2, S \subset \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{C}})$$

over quasi-stable rational 2-marked curves that satisfy the following conditions. The components of \mathcal{C} on which the corresponding quotient is torsion-free are rational and correspond to the edges of Γ ; the restriction of S to any such component corresponds to a morphism to \mathbb{P}^{n-1} of the opposite degree to that of the subsheaf. Furthermore, if $e = \{v_1, v_2\}$ is an edge, the corresponding morphism f_e is a degree- $\mathfrak{d}(e)$ cover of the line

$$\mathbb{P}_{\mu(v_1), \mu(v_2)}^1 \subset \mathbb{P}^{n-1}$$

passing through the fixed points $P_{\mu(v_1)}$ and $P_{\mu(v_2)}$; it is ramified only over $P_{\mu(v_1)}$ and $P_{\mu(v_2)}$. In particular, f_e is unique up to isomorphism. The remaining components of \mathcal{C} are indexed by the vertices $v \in \text{Ver}$ with $\mathfrak{d}(v) \in \mathbb{Z}^+$. The restriction of S to such a component \mathcal{C}_v of \mathcal{C} (or possibly a connected union of irreducible components) is a subsheaf of the trivial subsheaf $P_{\mu(v)} \subset \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{C}_v}$ of degree $-\mathfrak{d}(v)$; thus, the induced morphism takes \mathcal{C}_v to the fixed point $P_{\mu(v)} \in \mathbb{P}^{n-1}$. Each such component \mathcal{C}_v also carries two distinguished marked points, corresponding to the nodes and/or the marked points of \mathcal{C} ; if neither of the marked points of \mathcal{C} lies on \mathcal{C}_v , we denote the marked point corresponding to the node of \mathcal{C}_v separating \mathcal{C}_v from the first marked point by 1 and the other



Figure 2: The subgraphs corresponding to the edges of the last graph in Figure 1.

marked point by 2. Thus, as stacks,

$$\begin{aligned} Q_\Gamma &\approx \prod_{\substack{v \in \text{Ver} \\ \mathfrak{d}(v) > 0}} \overline{Q}_{0,2}(\mathbb{P}^0, \mathfrak{d}(v)) \times \prod_{e \in \text{Edg}} Q_{\Gamma_e} \approx \prod_{\substack{v \in \text{Ver} \\ \mathfrak{d}(v) > 0}} \overline{\mathcal{M}}_{0,2|\mathfrak{d}(v)}/\mathbb{S}_{\mathfrak{d}(v)} \times \prod_{e \in \text{Edg}} Q_{\Gamma_e} \\ &\approx \left(\prod_{\substack{v \in \text{Ver} \\ \mathfrak{d}(v) > 0}} \overline{\mathcal{M}}_{0,2|\mathfrak{d}(v)}/\mathbb{S}_{\mathfrak{d}(v)} \right) / \prod_{e \in \text{Edg}} \mathbb{Z}_{\mathfrak{d}(e)}, \end{aligned} \quad (6.3)$$

with each cyclic group $\mathbb{Z}_{\mathfrak{d}(e)}$ acting trivially. For example, in the case of the last diagram in Figure 1,

$$Q_\Gamma \approx \left(\overline{\mathcal{M}}_{0,2|d_0}/\mathbb{S}_{d_0} \times \overline{\mathcal{M}}_{0,2|5}/\mathbb{S}_5 \right) / \mathbb{Z}_d \times \mathbb{Z}_{d'}$$

is a fixed locus in $\overline{Q}_{0,2}(\mathbb{P}^{n-1}, d_0 + 5 + d + d')$.

If Γ is a decorated strand as above and $e \in \text{Edg}$, let

$$\pi_e : Q_\Gamma \longrightarrow Q_{\Gamma_e} \subset \overline{Q}_{0,2}(\mathbb{P}^{n-1}, \mathfrak{d}(e))$$

be the projection in the decomposition (6.3). Similarly, for each $v \in \text{Ver}$ such that $\mathfrak{d}(v) > 0$, let

$$\pi_v : Q_\Gamma \longrightarrow \overline{\mathcal{M}}_{0,2|\mathfrak{d}(v)}/\mathbb{S}_{\mathfrak{d}(v)}$$

be the corresponding projection. If $e = \{v_1, v_2\} \in \text{Edg}$ with $v_1 < v_2$, let

$$\omega_{e;v_1} = -\pi_e^* \psi_1, \omega_{e;v_2} = -\pi_e^* \psi_2, \psi_{v_1;e} = \pi_{v_1}^* \psi_2, \psi_{e;v_2} = \pi_{v_2}^* \psi_1 \in H^2(Q_\Gamma).$$

By [9, Section 27.2],

$$\omega_{e;v_i} = \frac{\alpha_{\mu(v_i)} - \alpha_{\mu(v_{3-i})}}{\mathfrak{d}(e)} \quad i = 1, 2. \quad (6.4)$$

For each $v \in \text{Ver} - \{v_{\min}\}$, let $e_-(v) = \{v_-, v\} \in \text{Edg}$ denote the edge with $v_- < v$; for each $v \in \text{Ver} - \{v_{\max}\}$, let $e_+(v) = \{v, v_+\} \in \text{Edg}$ denote the edge with $v < v_+$. By [13, Section 7.4], the euler class of the normal bundle of Q_Γ in $\overline{Q}_{0,2}(\mathbb{P}^{n-1}, |\Gamma|)$ is described by

$$\begin{aligned} \frac{\mathbf{e}(\mathcal{N}Q_\Gamma)}{\mathbf{e}(T_{\mu(v_{\min})}\mathbb{P}^{n-1})} &= \prod_{\substack{v \in \text{Ver} \\ \mathfrak{d}(v) > 0}} \prod_{k \neq \mu(v)} \pi_v^* \mathbf{e}(\mathcal{V}'_{1;\mathfrak{d}(v)}(\alpha_{\mu(v)} - \alpha_k)) \prod_{e \in \text{Edg}} \pi_e^* \mathbf{e}(H^0(f_e^* T\mathbb{P}^n \otimes \mathcal{O}(-y_1))/\mathbb{C}) \\ &\times \prod_{\substack{v \in \text{Ver} - v_{\min} - v_{\max} \\ \mathfrak{d}(v) = 0}} (\omega_{e_-(v);v} + \omega_{e_+(v);v}) \prod_{\substack{v \in \text{Ver} - v_{\min} \\ \mathfrak{d}(v) > 0}} (\omega_{e_-(v);v} - \psi_{v;e_-(v)}) \prod_{\substack{v \in \text{Ver} - v_{\max} \\ \mathfrak{d}(v) > 0}} (\omega_{e_+(v);v} - \psi_{v;e_+(v)}), \end{aligned} \quad (6.5)$$

where $\mathbb{C} \subset H^0(f_e^* T\mathbb{P}^n \otimes \mathcal{O}(-y_1))$ denotes the trivial \mathbb{T} -representation. The terms on the first line correspond to the deformations of the sheaf without changing the domain, while the terms on the second line correspond to the deformations of the domain. By (1.2) and (4.7),

$$\mathbf{e}(\mathcal{V}'_{n;\mathbf{a}})|_{Q_\Gamma} = \prod_{\substack{v \in \text{Ver} \\ \mathfrak{d}(v) > 0}} \pi_v^* \mathbf{e}(\mathcal{V}'_{\mathbf{a};\mathfrak{d}(v)}(\alpha_{\mu(v)})) \cdot \prod_{e \in \text{Edg}} \pi_e^* \mathbf{e}(\mathcal{V}_{n;\mathbf{a}}^{\mathfrak{d}(e)}). \quad (6.6)$$

By [9, Section 27.2],

$$\int_{Q_{\Gamma_e}} \frac{\mathbf{e}(\mathcal{V}'_{n;\mathbf{a}})^{\mathfrak{d}(e)}}{\mathbf{e}(H^0(f_e^* T\mathbb{P}^n \otimes \mathcal{O}(-y_1))/\mathbb{C})} = \mathfrak{C}_{\mu(v_1)}^{\mu(v_2)}(\mathfrak{d}(e)) \quad \forall e = \{v_1, v_2\} \text{ with } v_1 < v_2, \quad (6.7)$$

where $\mathfrak{C}_{\mu(v_1)}^{\mu(v_2)}(\mathfrak{d}(e))$ is as given by (5.6).

Proposition 6.1 is proved by applying the localization theorem to

$$\mathcal{Z}_{n;\mathbf{a}}(x = \alpha_i, \hbar, q) = 1 + \sum_{d=1}^{\infty} q^d \int_{\overline{Q}_{0,2}(\mathbb{P}^{n-1}, d)} \frac{\mathbf{e}(\mathcal{V}'_{n;d}) \text{ev}_1^* \phi_i}{\hbar - \psi_1} \in \mathbb{Q}_{\alpha}[[\hbar^{-1}, q]], \quad (6.8)$$

where ϕ_i is the equivariant Poincare dual of the fixed point $P_i \in \mathbb{P}^{n-1}$; see (3.4). Since $\phi_i|_{P_j} = 0$ unless $j = i$, a decorated strand as in (6.1) contributes to (6.8) only if the first marked point is attached to a vertex labeled i , i.e. $\mu(v_{\min}) = i$ for the smallest element $v_{\min} \in \text{Ver}$. We show that, just as for Givental's J -function, the (d, j) -summand in (5.4), i.e.

$$\mathcal{Z}_{n;\mathbf{a}}(\alpha_j, (\alpha_j - \alpha_i)/d, q),$$

is the sum over all strands such that $\mu(v_{\min}) = i$, i.e. the first marked point is mapped to the fixed point $P_i \in \mathbb{P}^{n-1}$, v_{\min} is a bivalent vertex, i.e. $\mathfrak{d}(v_{\min}) = 0$, the only edge leaving this vertex is labeled d , and the other vertex of this edge is labeled j . We also show that the first sum on the right-hand side of (5.4) is 1 (for the degree 0 term) plus the sum over all strands such that $\mu(v_{\min}) = i$ and $\mathfrak{d}(v_{\min}) > 0$.

If Γ is a decorated strand with $\mu(v_{\min}) = i$ as above,

$$\text{ev}_1^* \phi_i|_{Q_{\Gamma}} = \prod_{k \neq i} (\alpha_i - \alpha_k) = \mathbf{e}(T_{\mu(v_{\min})} \mathbb{P}^{n-1}). \quad (6.9)$$

Suppose in addition that $\mathfrak{d}(v_{\min}) = 0$. Let $v_1 \equiv (v_{\min})_+$ be the immediate successor of v_{\min} in Γ and $e_1 = \{v_{\min}, v_1\}$ be the edge leaving v_{\min} . If $|\text{Edg}| > 1$ or $\mathfrak{d}(v_1) > 0$ (i.e. Γ is not as in the first diagram in Figure 1), we break Γ at v_1 into two “sub-strands”:

- (i) $\Gamma_1 = \Gamma_{e_1}$ consisting of the vertices $v_{\min} < v_1$, the edge $\{v_{\min}, v_1\}$, and the \mathfrak{d} -value of 0 at both vertices;
- (ii) Γ_2 consisting all vertices and edges of Γ , other than the vertex v_{\min} and the edge $\{v_{\min}, v_1\}$;

see Figure 3. By (6.3),

$$Q_{\Gamma} \approx Q_{\Gamma_1} \times Q_{\Gamma_2}.$$

Let $\pi_1, \pi_2: Q_{\Gamma} \longrightarrow Q_{\Gamma_1}, Q_{\Gamma_2}$ be the two component projection maps. By (6.6) and (6.5),

$$\begin{aligned} \mathbf{e}(\mathcal{V}'_{n;\mathbf{a}})^{(|\Gamma|)}|_{Q_{\Gamma}} &= \pi_1^* \mathbf{e}(\mathcal{V}'_{n;\mathbf{a}})^{(|\Gamma_1|)} \cdot \pi_2^* \mathbf{e}(\mathcal{V}'_{n;\mathbf{a}})^{(|\Gamma_2|)}, \\ \frac{\mathbf{e}(\mathcal{N}Q_{\Gamma})}{\mathbf{e}(T_{P_i} \mathbb{P}^{n-1})} &= \pi_1^* \left(\frac{\mathbf{e}(\mathcal{N}Q_{\Gamma_1})}{\mathbf{e}(T_{P_i} \mathbb{P}^{n-1})} \right) \cdot \pi_2^* \left(\frac{\mathbf{e}(\mathcal{N}Q_{\Gamma_2})}{\mathbf{e}(T_{P_{\mu(v_1)}} \mathbb{P}^{n-1})} \right) \cdot (\omega_{e;v_1} - \pi_2^* \psi_1). \end{aligned}$$

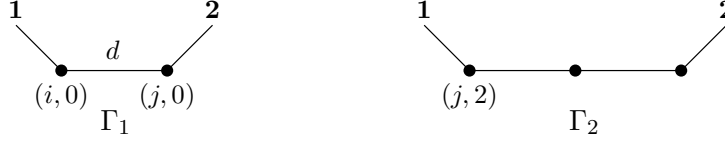


Figure 3: The two sub-strands of the second strand in Figure 1.

Combining this with (6.4), (6.7), and (6.9), we find that

$$\begin{aligned}
 q^{|\Gamma|} \int_{Q_\Gamma} \frac{\mathbf{e}(\mathcal{V}'_{n;\mathbf{a}}(|\Gamma|)) \text{ev}_1^* \phi_i}{\hbar - \psi_1} \Big|_{Q_\Gamma} \frac{1}{\mathbf{e}(NQ_\Gamma)} \\
 = \frac{\mathfrak{C}_i^{\mu(v_1)}(\mathfrak{d}(e_1)) q^{\mathfrak{d}(e_1)}}{\hbar - \frac{\alpha_{\mu(v_1)} - \alpha_i}{\mathfrak{d}(e_1)}} \cdot \left(q^{|\Gamma_2|} \left\{ \int_{Q_{\Gamma_2}} \frac{\mathbf{e}(\mathcal{V}'_{n;\mathbf{a}}(|\Gamma_2|)) \text{ev}_1^* \phi_{\mu(v_1)}}{\hbar - \psi_1} \frac{1}{\mathbf{e}(NQ_{\Gamma_2})} \right\} \Big|_{\hbar = \frac{\alpha_{\mu(v_1)} - \alpha_i}{\mathfrak{d}(e_1)}} \right). \quad (6.10)
 \end{aligned}$$

By (6.8) with i replaced by j and the localization formula (3.8), the sum of the factors over all possibilities for Γ_2 , with Γ_1 held fixed is

$$\mathcal{Z}(\alpha_{\mu(v_1)}, (\alpha_{\mu(v_1)} - \alpha_i)/\mathfrak{d}(e_1), q) - 1.$$

On the other hand, the contribution of the graph $\Gamma_{i\mu(v_1)}(\mathfrak{d}(e_1))$ as in the first diagram in Figure 1 is precisely the first factor on the right-hand side of (6.10). Thus, the contribution to (6.8) from all strands Γ such that $\mu(v_1) = j$ and $\mathfrak{d}(e_1) = d$ is

$$\frac{\mathfrak{C}_i^j(d) q^d}{\hbar - \frac{\alpha_j - \alpha_i}{d}} \mathcal{Z}(\alpha_j, (\alpha_j - \alpha_i)/d, q),$$

i.e. the (d, j) -summand in the recursion (5.4) for $\mathcal{Z}_{n;\mathbf{a}}$.

Suppose next that Γ is a strand such that $\mu(v) = i$ and $\mathfrak{d}(v_{\min}) > 0$. If $|\text{Ver}| > 1$ (i.e. Γ is not as in the first diagram in Figure 4), we break Γ at v_{\min} into two “sub-strands”:

- (i) Γ_0 consisting of the vertex $\{v_{\min}\}$ only, with the same μ and \mathfrak{d} -values as in Γ ;
- (ii) Γ_c consisting all vertices and edges of Γ , but with the \mathfrak{d} -value of v_{\min} replaced by 0;

see Figure 4. By (6.3),

$$Q_\Gamma \approx Q_{\Gamma_0} \times Q_{\Gamma_c} = (\overline{\mathcal{M}}_{0,2|\mathfrak{d}(v_{\min})}/\mathbb{S}_{\mathfrak{d}(v_{\min})}) \times Q_{\Gamma_c}; \quad (6.11)$$

if $|\text{Ver}| = 1$, this decomposition holds with $Q_{\Gamma_c} \equiv \{pt\}$ and $\mathfrak{d}(v_{\min}) = |\Gamma|$. Let π_0, π_c be the two component projection maps in (6.11). Since

$$\psi_1|_{Q_\Gamma} = \pi_0^* \psi_1,$$

\mathbb{T} acts trivially on $\overline{\mathcal{M}}_{0,2|\mathfrak{d}(v_{\min})}$,

$$\psi_1 = 1 \times \psi_1 \in H_{\mathbb{T}}^*(\overline{\mathcal{M}}_{0,2|\mathfrak{d}(v_{\min})}) = H_{\mathbb{T}}^* \otimes H^*(\overline{\mathcal{M}}_{0,2|\mathfrak{d}(v_{\min})}),$$

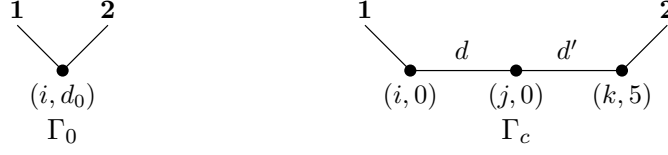


Figure 4: The two sub-strands of the last strand in Figure 1.

i.e. \mathbb{T} acts trivial on the universal cotangent line bundle for the first marked point on $\overline{\mathcal{M}}_{0,2|\mathfrak{d}(v_{\min})}$, and the dimension of $\overline{\mathcal{M}}_{0,2|\mathfrak{d}(v_{\min})}$ is $\mathfrak{d}(v_{\min}) - 1$,

$$\frac{1}{\hbar - \psi_1}|_{Q_\Gamma} = \sum_{r=0}^{\mathfrak{d}(v_{\min})-1} \hbar^{-(r+1)} \pi_0^* \psi_1^r.$$

Since $|\mathfrak{d}(v_{\min})| \leq |\Gamma|$ and Γ contributes to the coefficient of $q^{|\Gamma|}$ in (6.8), it follows that $\mathcal{Z}_{n;\mathbf{a}}$ satisfies (5.4) with $\mathcal{F} = \mathcal{Z}_{n;\mathbf{a}}$, $C_i^j(d) = \mathfrak{C}_i^j(d)$, $N_d = d$, $\mathcal{Z}_i^r(d) = 0$ for $r \in \mathbb{Z}^+$, and $\mathcal{Z}_i^0(d) = \delta_{0d}$. In particular, $\mathcal{Z}_{n;\mathbf{a}}$ is \mathfrak{C} -recursive.

It remains to verify the last identity in Proposition 6.1. We continue with the notation as in the previous paragraph. If $|\text{Ver}| = 1$, the second factor in (6.11) is trivial; in this case, (6.6) and (6.5) immediately give

$$q^{|\Gamma|} \int_{Q_\Gamma} \frac{\mathbf{e}(\mathcal{V}'_{n;\mathbf{a}}(|\Gamma|)) \text{ev}_1^* \phi_i}{\hbar - \psi_1} \Big|_{Q_\Gamma} \frac{1}{\mathbf{e}(NQ_\Gamma)} = \sum_{r=0}^{|\Gamma|-1} \hbar^{-(r+1)} \frac{q^{|\Gamma|}}{(|\Gamma|)!} \int_{\overline{\mathcal{M}}_{0,2||\Gamma|}} \frac{\mathbf{e}(\mathcal{V}'_{\mathbf{a};|\Gamma|}(\alpha_i)) \psi_1^r}{\prod_{k \neq i} \mathbf{e}(\mathcal{V}'_{1;|\Gamma|}(\alpha_i - \alpha_k))}. \quad (6.12)$$

Suppose next that $|\text{Ver}| > 1$. By (6.6) and (6.5),

$$\begin{aligned} \mathbf{e}(\mathcal{V}'_{n;\mathbf{a}}(|\Gamma|)) \Big|_{Q_\Gamma} &= \pi_0^* \mathbf{e}(\mathcal{V}'_{\mathbf{a};|\Gamma_0|}(\alpha_i)) \cdot \pi_c^* \mathbf{e}(\mathcal{V}'_{n;\mathbf{a}}(|\Gamma_c|)), \\ \frac{\mathbf{e}(NQ_\Gamma)}{\mathbf{e}(T_{P_i} \mathbb{P}^{n-1})} &= \pi_0^* \prod_{k \neq i} \mathbf{e}(\mathcal{V}'_{1;|\Gamma_0|}(\alpha_i - \alpha_k)) \cdot \pi_c^* \left(\frac{\mathbf{e}(NQ_{\Gamma_c})}{\mathbf{e}(T_{P_i} \mathbb{P}^{n-1})} \right) \cdot (\omega_{e;v_{\min}} - \pi_0^* \psi_2), \end{aligned}$$

where e_1 is the edge leaving v_{\min} . Combining this with (6.9), we find that

$$\begin{aligned} q^{|\Gamma|} \int_{Q_\Gamma} \frac{\mathbf{e}(\mathcal{V}'_{n;\mathbf{a}}(|\Gamma|)) \text{ev}_1^* \phi_i}{\hbar - \psi_1} \Big|_{Q_\Gamma} \frac{1}{\mathbf{e}(NQ_\Gamma)} &= \sum_{r=0}^{d_0-1} \sum_{b=0}^{d_0-1-r} \hbar^{-(r+1)} \left(\frac{q^{d_0}}{d_0!} \int_{\overline{\mathcal{M}}_{0,2|d_0}} \frac{\mathbf{e}(\mathcal{V}'_{\mathbf{a};d_0}(\alpha_i)) \psi_1^r \psi_2^b}{\prod_{k \neq i} \mathbf{e}(\mathcal{V}'_{1;d_0}(\alpha_i - \alpha_k))} \right. \\ &\quad \left. \times (-1)^{b+1} q^{|\Gamma_c|} \int_{Q_{\Gamma_c}} \psi_1^{-(b+1)} \frac{\mathbf{e}(\mathcal{V}'_{n;\mathbf{a}}(|\Gamma_c|)) \text{ev}_1^* \phi_i}{\mathbf{e}(NQ_{\Gamma_c})} \right), \end{aligned}$$

where $d_0 = \mathfrak{d}(v_{\min}) = |\Gamma_0|$.

We now sum up the last factor above over all possibilities for Γ_c with $|\Gamma_c| > 0$ by decomposing Γ_c into sub-strands $\Gamma_1 = \Gamma_{ij}(d)$, for some $j \in [n] - i$ and $d \in \mathbb{Z}^+$, and Γ_2 , as in the case $\mathfrak{d}(v_{\min}) = 0$ above.

If $\Gamma_c \neq \Gamma_1$, similarly to (6.10) we obtain

$$q^{|\Gamma_c|} \int_{Q_{\Gamma_c}} \psi_1^{-(b+1)} \frac{\mathbf{e}(\mathcal{V}'_{n;\mathbf{a}}(|\Gamma_c|)) \text{ev}_1^* \phi_i}{\mathbf{e}(\mathcal{N}Q_{\Gamma_c})} = \mathfrak{e}_i^{\mu(v_1)}(\mathfrak{d}(e_1)) q^{\mathfrak{d}(e_1)} \left(\frac{\alpha_{\mu(v_1)} - \alpha_i}{\mathfrak{d}(e_1)} \right)^{-(b+1)} \\ \times \left(q^{|\Gamma_2|} \left\{ \int_{Q_{\Gamma_2}} \frac{\mathbf{e}(\mathcal{V}'_{n;\mathbf{a}}(|\Gamma_2|)) \text{ev}_1^* \phi_{\mu(v_1)}}{\hbar - \psi_1} \frac{1}{\mathbf{e}(\mathcal{N}Q_{\Gamma_2})} \right\} \Big|_{\hbar = \frac{\alpha_{\mu(v_1)} - \alpha_i}{\mathfrak{d}(e_1)}} \right).$$

The sum of the last factor above over all possibilities for Γ_2 , including the empty case when this factor is taken to be 1 for the equality to hold, with Γ_1 held fixed, is

$$\mathcal{Z}(\alpha_{\mu(v_1)}, (\alpha_{\mu(v_1)} - \alpha_i) / \mathfrak{d}(e_1), q),$$

as before. Comparing this with the recursion (5.4) for $\mathcal{Z}_{n;\mathbf{a}}$, we conclude

$$\sum_{\substack{\Gamma_c, |\Gamma_c| > 0 \\ \mu(v_1) = j, \mathfrak{d}(e_1) = d}} q^{|\Gamma_c|} \int_{Q_{\Gamma_c}} \psi_1^{-(b+1)} \frac{\mathbf{e}(\mathcal{V}'_{n;\mathbf{a}}(|\Gamma_c|)) \text{ev}_1^* \phi_i}{\mathbf{e}(\mathcal{N}Q_{\Gamma_c})} = \mathfrak{R}_{\hbar = \frac{\alpha_j - \alpha_i}{d}} \{ \hbar^{-(b+1)} \mathcal{Z}(\alpha_i, \hbar, q) \}.$$

Thus, by the recursion (5.4) and the Residue Theorem on S^2 ,

$$\sum_{\Gamma_c, |\Gamma_c| > 0} q^{|\Gamma_c|} \int_{Q_{\Gamma_c}} \psi_1^{-(b+1)} \frac{\mathbf{e}(\mathcal{V}'_{n;\mathbf{a}}(|\Gamma_c|)) \text{ev}_1^* \phi_i}{\mathbf{e}(\mathcal{N}Q_{\Gamma_c})} = - \mathfrak{R}_{\hbar=0, \infty} \{ \hbar^{-(b+1)} \mathcal{Z}(\alpha_i, \hbar, q) \} \\ = - \mathfrak{R}_{\hbar=0} \{ \hbar^{-(b+1)} \mathcal{Z}(\alpha_i, \hbar, q) \} + \delta_{0b}.$$

Combining this with (6.12), we conclude that

$$\sum_{\Gamma, \mathfrak{d}(v_{\min}) > 0} q^{|\Gamma|} \int_{Q_{\Gamma}} \frac{\mathbf{e}(\mathcal{V}'_{n;\mathbf{a}}(|\Gamma|)) \text{ev}_1^* \phi_i}{\hbar - \psi_1} \Big|_{Q_{\Gamma}} \frac{1}{\mathbf{e}(\mathcal{N}Q_{\Gamma})} \\ = \sum_{d=1}^{\infty} \frac{q^d}{d!} \sum_{r=0}^{d-1} \hbar^{-(r+1)} \sum_{b=0}^{d-1-r} \left(\left(\int_{\overline{\mathcal{M}}_{0,2|d}} \frac{\mathbf{e}(\mathcal{V}'_{\mathbf{a};d}(\alpha_i)) \psi_1^r \psi_2^b}{\prod_{k \neq i} \mathbf{e}(\mathcal{V}'_{1;d}(\alpha_i - \alpha_k))} \right) \mathfrak{R}_{\hbar=0} \left\{ \frac{(-1)^b}{\hbar^{b+1}} \mathcal{Z}(\alpha_i, \hbar, q) \right\} \right).$$

This concludes the proof of Proposition 6.1.

In the case of products of projective spaces and concavex sheaves (1.7), we need analogues of (4.6) and (4.7) for every pair of tuples

$$\mathbf{d} \equiv (d_1, \dots, d_p) \in (\mathbb{Z}^{\geq 0})^p - 0, \quad \beta = (\beta_1, \dots, \beta_p) \in H_{\mathbb{T}}^2.$$

Thus, we define sheaves $\mathcal{S}_1^*, \dots, \mathcal{S}_p^*$ over the universal curve $\mathcal{U} \rightarrow \overline{\mathcal{M}}_{0,2||\mathbf{d}|}$ by

$$\mathcal{S}_1^* \equiv \mathcal{O}_{\mathcal{U}}(\sigma_1 + \dots + \sigma_{d_1}), \mathcal{S}_2^* \equiv \mathcal{O}_{\mathcal{U}}(\sigma_{d_1+1} + \dots + \sigma_{d_1+d_2}), \dots \rightarrow \mathcal{U}$$

and denote by $\mathcal{S}_i^*(\beta_i)$, with $i=1, \dots, p$, the sheaves such that

$$\mathbf{e}(\mathcal{S}_i^*(\beta_i)) = \beta_i \times 1 + 1 \times e(\mathcal{S}_i^*) \in H_{\mathbb{T}}^*(\mathcal{U}) = H_{\mathbb{T}}^* \otimes H^*(\mathcal{U}).$$

Similarly to (4.7), let

$$\begin{aligned} \mathcal{V}'_{\mathbf{a};\mathbf{d}}(\beta) &= \bigoplus_{a_{k;1} > 0} R^0 \pi_* (\mathcal{S}_1^*(\beta_1)^{a_{k;1}} \otimes \dots \otimes \mathcal{S}_1^*(\beta_p)^{a_{k;p}} - \sigma_1) \\ &\quad \oplus \bigoplus_{a_{k;1} < 0} R^1 \pi_* (\mathcal{S}_1^*(\beta_1)^{a_{k;1}} \otimes \dots \otimes \mathcal{S}_1^*(\beta_p)^{a_{k;p}} - \sigma_1) \longrightarrow \overline{\mathcal{M}}_{0,2||\mathbf{d}|}. \end{aligned}$$

The fixed points of the \mathbb{T} -action on $\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_s-1}$ are

$$P_{i_1 \dots i_p} \equiv P_{i_1} \times \dots \times P_{i_p}, \quad i_s \in [n_s];$$

thus, the function μ on vertices now takes values in the tuples (i_1, \dots, i_p) . The function \mathfrak{d} on vertices now takes values in $(\mathbb{Z}^{\geq 0})^p$, with the space $\overline{\mathcal{M}}_{0,2|\mathfrak{d}(v)|}/\mathbb{S}_{\mathfrak{d}(v)}$ above replaced by

$$\overline{\mathcal{M}}_{0,2|\mathfrak{d}_1(v)+\dots+\mathfrak{d}_p(v)|}/\mathbb{S}_{\mathfrak{d}_1(v)} \times \dots \times \mathbb{S}_{\mathfrak{d}_p(v)},$$

in light of (2.7). The \mathbb{T} -fixed curves are the lines between the points $P_{i_1 \dots i_p}$ and $P_{j_1 \dots j_p}$ such that

$$|\{s \in [p] : i_s \neq j_s\}| = 1;$$

thus, the vertices of any edge now differ by precisely one of the indices (i_1, \dots, i_p) , with the ω -classes in (6.4) described by the difference in the weights of this index. The strands with $\mathfrak{d}(v_{\min}) = 0$ now give rise to a triple sum, with the summation index $s \in [p]$ on the outer sum indicating which of the indices (i_1, \dots, i_p) changes. The computation of the contribution from the strands with $\mathfrak{d}(v_{\min}) > 0$ proceeds exactly as above, but the denominator in the integrand for $\overline{\mathcal{M}}_{0,2|d_0|}$ above is replaced by the product of factors corresponding to each of the p factors. This results in a similar formula for the secondary coefficients $\mathcal{Z}_{i_1 \dots i_p}^r$ in (5.4):

$$\begin{aligned} \sum_{(d_1, \dots, d_p) \in (\mathbb{Z}^{\geq 0}) - 0} \mathcal{Z}_{i_1 \dots i_p}^r(d_1, \dots, d_p) q_1^{d_1} \dots q_p^{d_p} &= \sum_{\mathbf{d} \in (\mathbb{Z}^{\geq 0}) - 0} \frac{q_1^{d_1} \dots q_p^{d_p}}{d_1! \dots d_p!} \sum_{b=0}^{|\mathbf{d}|+r} \\ &\quad \left(\left(\int_{\overline{\mathcal{M}}_{0,2||\mathbf{d}|}} \frac{\mathbf{e}(\mathcal{V}'_{\mathbf{a};\mathbf{d}}(\alpha_{i_1}, \dots, \alpha_{i_p})) \psi_1^{-r-1} \psi_2^b}{\prod_{s=1}^p \prod_{k \neq i_s} \mathbf{e}(\mathcal{V}'_{e_s; d_s}(\alpha_{s; i_s} - \alpha_{s; k}))} \right) \Re \left\{ \frac{(-1)^b}{\hbar^{b+1}} \mathcal{Z}_{n; \mathbf{a}}(\alpha_{i_1}, \dots, \alpha_{i_p}, \hbar, q_1, \dots, q_p) \right\} \right), \end{aligned} \quad (6.13)$$

whenever $r \in \mathbb{Z}^-$ and $i_s \in [n_s]$, if $e_s \in (\mathbb{Z}^+)^p$ is the s -th coordinated vector.

7 Polynomiality for stable quotients

In this section, we adopt the argument in [9, Section 30.2], showing that the equivariant version of Givental's J -function satisfies the self-polynomiality condition of Definition 5.2, to show that the equivariant stable quotients analogue of Givental's J -function, the power series $\mathcal{Z}_{n; \mathbf{a}}$ defined by (4.1), also satisfies the self-polynomiality condition. Proposition 7.1 is an immediate consequence of Lemma 7.2 below, which provides a geometric description of the power series $\mathcal{Z}_{n; \mathbf{a}}$.

Proposition 7.1. *If $l \in \mathbb{Z}^{\geq 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$, the power series $\mathcal{Z}_{n; \mathbf{a}}(\mathbf{x}, \hbar, q)$ satisfies the self-polynomiality condition.*

The proof involves applying the classical localization theorem [1] with $(n+1)$ -torus

$$\widetilde{\mathbb{T}} \equiv \mathbb{C}^* \times \mathbb{T},$$

where $\mathbb{T} = (\mathbb{C}^*)^n$ as before. We denote the weight of the standard action of the one-torus \mathbb{C}^* on \mathbb{C} by \hbar . Thus, by Section 3,

$$H_{\mathbb{C}^*}^* \approx \mathbb{Q}[\hbar], \quad H_{\widetilde{\mathbb{T}}}^* \approx \mathbb{Q}[\hbar, \alpha_1, \dots, \alpha_n] \quad \implies \quad \mathcal{H}_{\widetilde{\mathbb{T}}}^* \approx \mathbb{Q}_\alpha(\hbar).$$

Throughout this section, $V = \mathbb{C} \oplus \mathbb{C}$ denotes the representation of \mathbb{C}^* with the weights 0 and $-\hbar$. The induced action on $\mathbb{P}V$ has two fixed points:

$$q_1 \equiv [1, 0], \quad q_2 \equiv [0, 1].$$

With $\gamma_1 \longrightarrow \mathbb{P}V$ denoting the tautological line bundle,

$$\mathbf{e}(\gamma_1^*)|_{q_1} = 0, \quad \mathbf{e}(\gamma_1^*)|_{q_2} = -\hbar, \quad \mathbf{e}(T_{q_1}\mathbb{P}V) = \hbar, \quad \mathbf{e}(T_{q_2}\mathbb{P}V) = -\hbar; \quad (7.1)$$

this follows from our definition of the weights in Section 3.

For each $d \in \mathbb{Z}^{\geq 0}$, the action of $\widetilde{\mathbb{T}}$ on $\mathbb{C}^n \otimes \text{Sym}^d V^*$ induces an action on

$$\overline{\mathfrak{X}}_d \equiv \mathbb{P}(\mathbb{C}^n \otimes \text{Sym}^d V^*).$$

It has $(d+1)n$ fixed points:

$$P_i(r) \equiv [\tilde{P}_i \otimes u^{d-r} v^r], \quad i \in [n], \quad r \in \{0\} \cup [d],$$

if (u, v) are the standard coordinates on V and $\tilde{P}_i \in \mathbb{C}^n$ is the i -th coordinate vector (so that $[\tilde{P}_i] = P_i \in \mathbb{P}^{n-1}$). Let

$$\Omega \equiv \mathbf{e}(\gamma^*) \in H_{\widetilde{\mathbb{T}}}^*(\overline{\mathfrak{X}}_d)$$

denote the equivariant hyperplane class.

For all $i \in [n]$ and $r \in \{0\} \cup [d]$,

$$\Omega|_{P_i(r)} = \alpha_i + r\hbar, \quad \mathbf{e}(T_{P_i(r)}\overline{\mathfrak{X}}_d) = \left\{ \prod_{s=0}^{s=d} \prod_{\substack{k=1 \\ (s,k) \neq (r,i)}}^{k=n} (\Omega - \alpha_k - s\hbar) \right\} \Big|_{\Omega=\alpha_i+r\hbar}.^6 \quad (7.2)$$

Since

$$B\overline{\mathfrak{X}}_d = \mathbb{P}(B(\mathbb{C}^n \otimes \text{Sym}^d V^*)) \longrightarrow B\widetilde{\mathbb{T}} \quad \text{and} \\ c(B(\mathbb{C}^n \otimes \text{Sym}^d V^*)) = \prod_{s=0}^{s=d} \prod_{k=1}^{k=n} (1 - (\alpha_k + s\hbar)) \in H^*(B\widetilde{\mathbb{T}}),^7$$

⁶The weight (i.e. negative first chern class) of the $\widetilde{\mathbb{T}}$ -action on the line $P_i(r) \subset \mathbb{C}^n \otimes \text{Sym}^d V^*$ is $\alpha_i + r\hbar$. The tangent bundle of $\overline{\mathfrak{X}}_d$ at $P_i(r)$ is the direct sum of the lines $P_i(r)^* \otimes P_k(s)$ with $(k, s) \neq (i, r)$.

the $\widetilde{\mathbb{T}}$ -equivariant cohomology of $\overline{\mathfrak{X}}_d$ is given by

$$\begin{aligned} H_{\widetilde{\mathbb{T}}}^*(\overline{\mathfrak{X}}_d) &\equiv H^*(B\overline{\mathfrak{X}}_d) = H^*(B\widetilde{\mathbb{T}})[\Omega] \Big/ \prod_{s=0}^{s=d} \prod_{k=1}^{k=n} (\Omega - (\alpha_k + s\hbar)) \\ &\approx \mathbb{Q}[\Omega, \hbar, \alpha_1, \dots, \alpha_n] \Big/ \prod_{s=0}^{s=d} \prod_{k=1}^{k=n} (\Omega - \alpha_k - s\hbar) \\ &\subset \mathbb{Q}_\alpha[\hbar, \Omega] \Big/ \prod_{s=0}^{s=d} \prod_{k=1}^{k=n} (\Omega - \alpha_s - s\hbar). \end{aligned}$$

In particular, every element of $H_{\widetilde{\mathbb{T}}}^*(\overline{\mathfrak{X}}_d)$ is a polynomial in Ω with coefficients in $\mathbb{Q}_\alpha[\hbar]$ of degree at most $(d+1)n-1$.

By [10, Lemma 2.6], there is a natural $\widetilde{\mathbb{T}}$ -equivariant morphism

$$\Theta: \overline{\mathfrak{M}}_{0,m}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1, d)) \longrightarrow \overline{\mathfrak{X}}_d.$$

A general element b of $\overline{\mathfrak{M}}_{0,m}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1, d))$ determines a morphism

$$(f, g): \mathbb{P}^1 \longrightarrow (\mathbb{P}V, \mathbb{P}^{n-1}),$$

up to an automorphism of the domain \mathbb{P}^1 . Thus, the morphism

$$g \circ f^{-1}: \mathbb{P}V \longrightarrow \mathbb{P}^{n-1}$$

is well-defined and determines an element $\Theta(b) \in \overline{\mathfrak{X}}_d$. Let

$$\begin{aligned} \mathfrak{X}_d &= \{b \in \overline{\mathfrak{M}}_{0,2}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1, d)) : \text{ev}_1(b) \in q_1 \times \mathbb{P}^{n-1}, \text{ev}_2(b) \in q_2 \times \mathbb{P}^{n-1}\}, \\ \mathfrak{X}'_d &= \{b' \in \overline{\mathcal{Q}}_{0,2}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1, d)) : \text{ev}_1(b') \in q_1 \times \mathbb{P}^{n-1}, \text{ev}_2(b') \in q_2 \times \mathbb{P}^{n-1}\}. \end{aligned} \quad (7.3)$$

Since the morphism to \mathbb{P}^1 corresponding to any element of $b' \in \mathfrak{X}'_d$ takes the two marked points to q_1 and q_2 , it is not constant. Thus, the restriction of the morphism Θ to \mathfrak{X}_d is constant along the fibers of the natural surjective morphism $c: \mathfrak{X}_d \longrightarrow \mathfrak{X}'_d$.⁸ It follows that the restriction of Θ to \mathfrak{X}_d descends via c to a morphism

$$\theta = \theta_d: \mathfrak{X}'_d \longrightarrow \overline{\mathfrak{X}}_d.$$

For $d > 0$, there is also a natural forgetful morphism

$$F: \overline{\mathcal{Q}}_{0,2}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1, d)) \longrightarrow \overline{\mathcal{Q}}_{0,2}(\mathbb{P}^{n-1}, d),$$

which drops the first sheaf in the pair and contracts one component of the domain if necessary. In the case $d=0$, we set

$$F^* \mathbf{e}(\mathcal{V}_{n;\mathbf{a}}^{(0)}) = \langle \mathbf{a} \rangle \text{ev}_1^*(1 \times \mathbf{x}^{\ell(\mathbf{a})}) \in H^*(\overline{\mathcal{Q}}_{0,2}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1, 0)));$$

⁷The vector space $\mathbb{C}^n \otimes \text{Sym}^d V^*$ is the direct sum of the one-dimensional representations $P_k(s)$ of $\widetilde{\mathbb{T}}$.

⁸For a stable map b , $\Theta(b)$ depends only on the restriction of b to the irreducible component $\mathcal{C}_{b;1}$ of its domain \mathcal{C}_b on which the degree of the map to \mathbb{P}^1 is not zero, the nodes of $\mathcal{C}_{b;1}$, and the degrees of the restrictions of b to the connected components of $\mathcal{C}_b - \mathcal{C}_{b;1}$. In contrast, $c(b)$ depends on the restriction of b to the minimal connected union (chain) of irreducible components \mathcal{C}'_b of its domain which contains the two marked points, the nodes of \mathcal{C}'_b , and the degrees of the restrictions of b to the connected components of $\mathcal{C}_b - \mathcal{C}'_b$. Whenever $b \in \mathfrak{X}_d$, $\mathcal{C}_{b;1} \subset \mathcal{C}'_b$. Thus, the restriction of Θ to \mathfrak{X}_d contracts everything that the restriction of c contracts.

this is used in Lemma 7.2 below.

Similarly to (1.2), for each $d \in \mathbb{Z}^+$ let

$$\mathcal{V}_{n;\mathbf{a}}^{(d)} = \bigoplus_{a_k > 0} R^0 \pi_* (\mathcal{S}^{*a_k}) \oplus \bigoplus_{a_k < 0} R^1 \pi_* (\mathcal{S}^{*a_k}) \longrightarrow \overline{Q}_{0,2}(\mathbb{P}^{n-1}, d).$$

From the usual short exact sequence for the restriction along σ_1 , we find that

$$\mathbf{e}(\mathcal{V}_{n;\mathbf{a}}^{(d)}) = \langle \mathbf{a} \rangle \text{ev}_1^* \mathbf{x}^{\ell(\mathbf{a})} \mathbf{e}(\mathcal{V}'_{n;\mathbf{a}}^{(d)}) \in H_{\mathbb{T}}^*(\overline{Q}_{0,2}(\mathbb{P}^{n-1}, d)). \quad (7.4)$$

Lemma 7.2. *If $l \in \mathbb{Z}^{\geq 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$, then*

$$\Phi_{\mathcal{Z}_{n;\mathbf{a}}}(\hbar, z, q) = \sum_{d=0}^{\infty} q^d \int_{\mathfrak{X}'_d} e^{(\theta^* \Omega)z} F^* \mathbf{e}(\mathcal{V}_{n;\mathbf{a}}^{(d)}) \in H_{\mathbb{T}}^*[[z, q]] \subset \mathbb{Q}_\alpha[\hbar][[z, q]]. \quad (7.5)$$

We prove Lemma 7.2 in the remainder of this section by applying the localization theorem of [1] to the $\widetilde{\mathbb{T}}$ -action on \mathfrak{X}'_d . We show that each fixed locus of the $\widetilde{\mathbb{T}}$ -action on \mathfrak{X}'_d contributing to the right-hand side of (7.5) corresponds to a pair (Γ_1, Γ_2) of decorated strands as in (6.1), with Γ_1 and Γ_2 contributing to $\mathcal{Z}_{n;\mathbf{a}}(\alpha_i, \hbar, qe^{\hbar z})$ and $\mathcal{Z}_{n;\mathbf{a}}(\alpha_i, -\hbar, q)$, respectively, for some $i \in [n]$.

Similarly to Section 6, the fixed loci of the $\widetilde{\mathbb{T}}$ -action on $\overline{Q}_{0,2}(\mathbb{P}V \times \mathbb{P}^{n-1}, (d', d))$ correspond to decorated strands Γ with 2 marked points at the opposite ends. The map \mathfrak{d} should now take values in pairs of nonnegative integers, indicating the degrees of the two subsheaves. The map μ should similarly take values in the pairs $(1, j)$ and $(2, j)$ with $j \in [n]$, indicating the fixed point, (q_1, P_j) or (q_2, P_j) , to which the vertex is mapped. The μ -values on consecutive vertices must differ by precisely one of the two components.

The situation for the $\widetilde{\mathbb{T}}$ -action on

$$\mathfrak{X}'_d \subset \overline{Q}_{0,2}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1, d))$$

is simpler, however. There is a unique edge of positive $\mathbb{P}V$ -degree; we draw it as a thick line in Figure 5. The first component of the value of \mathfrak{d} on all other edges and on all vertices must be 0; so we drop it. The first component of the value of μ on the vertices changes only when the thick edge is crossed. Thus, we drop the first components of the vertex labels as well, with the convention that these components are 1 on the left side of the thick edge and 2 on the right. In particular, the vertices to the left of the thick edge (including the left endpoint) lie in $q_1 \times \mathbb{P}^{n-1}$ and the vertices to its right lie in $q_2 \times \mathbb{P}^{n-1}$. Thus, by (7.3), the marked point 1 is attached to a vertex to the left of the thick edge and the marked point 2 is attached to a vertex to the right. Finally, the remaining, second component of μ takes the same value $i \in [n]$ on the two vertices of the thick edge.

Let \mathcal{A}_i denote the set of strands as above so that the μ -value on the two endpoints of the thick edge is labeled i ; see Figure 5. We break each strand $\Gamma \in \mathcal{A}_i$ into three sub-strands:

- (i) Γ_1 consisting of all vertices of Γ to the left of the thick edge, including its left vertex v_1 with its \mathfrak{d} -value, but in the opposite order, and a new marked point attached to v_1 ;

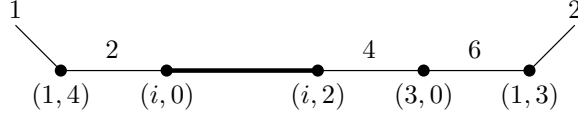


Figure 5: A strand representing a fixed locus in \mathfrak{X}'_d ; $i \neq 1, 3$

- (ii) Γ_0 consisting of the thick edge e_0 , its two vertices v_1 and v_2 , with \mathfrak{d} -values set to 0, and new marked points 1 and 2 attached to v_1 and v_2 , respectively;
- (iii) Γ_2 consisting of all vertices to the right of the thick edge, including its right vertex v_2 with its \mathfrak{d} -value, and a new marked point attached to v_2 ;

see Figure 6. From (6.3), we then obtain a splitting of the fixed locus in \mathfrak{X}_d corresponding to Γ :

$$Q_\Gamma \approx Q_{\Gamma_1} \times Q_{\Gamma_0} \times Q_{\Gamma_2} \subset \overline{Q}_{0,2}(\mathbb{P}^{n-1}, |\Gamma_1|) \times \overline{Q}_{0,2}(\mathbb{P}V, 1) \times \overline{Q}_{0,2}(\mathbb{P}^{n-1}, |\Gamma_2|). \quad (7.6)$$

The exceptional cases are $|\Gamma_1|=0$ and $|\Gamma_2|=0$; the above isomorphism then holds with the corresponding component replaced by a point.

Let π_1 , π_0 , and π_2 denote the three component projection maps in (7.6). By (7.4), (6.6), and (6.5),

$$\begin{aligned} F^* \mathbf{e}(\mathcal{V}_{n;\mathbf{a}}^{(|\Gamma|)})|_{Q_\Gamma} &= \langle \mathbf{a} \rangle \alpha_i^{\ell(\mathbf{a})} \cdot \pi_1^* \mathbf{e}(\mathcal{V}_{n;\mathbf{a}}^{(|\Gamma_1|)}) \cdot \pi_2^* \mathbf{e}(\mathcal{V}_{n;\mathbf{a}}^{(|\Gamma_2|)}), \\ \frac{\mathbf{e}(\mathcal{N}Q_\Gamma)}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})} &= \pi_1^* \left(\frac{\mathbf{e}(\mathcal{N}Q_{\Gamma_1})}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})} \right) \cdot \pi_2^* \left(\frac{\mathbf{e}(\mathcal{N}Q_{\Gamma_2})}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})} \right) \cdot (\omega_{e_0;v_1} - \pi_1^* \psi_1)(\omega_{e_0;v_2} - \pi_2^* \psi_1). \end{aligned} \quad (7.7)$$

Since Q_{Γ_0} consists of a degree 1 map, by the last two identities in (7.1)

$$\omega_{e_0;v_1} = \hbar, \quad \omega_{e_0;v_2} = -\hbar. \quad (7.8)$$

The morphism θ takes the locus Q_Γ to a fixed point $P_k(r) \in \overline{\mathfrak{X}}_d$. It is immediate that $k=i$. By continuity considerations, $r=|\Gamma_1|$. Thus, by the first identity in (7.2),

$$\theta^* \Omega|_{Q_\Gamma} = \alpha_i + |\Gamma_1| \hbar. \quad (7.9)$$

Combining (7.7)-(7.9), we obtain

$$\begin{aligned} q^{|\Gamma|} \int_{Q_\Gamma} \frac{e^{(\theta^* \Omega)z} F^* \mathbf{e}(\mathcal{V}_{n;\mathbf{a}}^{(|\Gamma|)})|_{Q_\Gamma}}{\mathbf{e}(\mathcal{N}Q_\Gamma)} &= \frac{\langle \mathbf{a} \rangle \alpha_i^{\ell(\mathbf{a})} e^{\alpha_i z}}{\prod_{k \neq i} (\alpha_i - \alpha_k)} \left\{ e^{|\Gamma_1| \hbar z} q^{|\Gamma_1|} \int_{Q_{\Gamma_1}} \frac{\mathbf{e}(\mathcal{V}_{n;\mathbf{a}}^{(|\Gamma_1|)}) \text{ev}_1^* \phi_i}{\hbar - \psi_1} \Big|_{Q_{\Gamma_1}} \frac{1}{\mathbf{e}(\mathcal{N}Q_{\Gamma_1})} \right\} \\ &\quad \times \left\{ q^{|\Gamma_2|} \int_{Q_{\Gamma_2}} \frac{\mathbf{e}(\mathcal{V}_{n;\mathbf{a}}^{(|\Gamma_2|)}) \text{ev}_1^* \phi_i}{(-\hbar) - \psi_1} \Big|_{Q_{\Gamma_2}} \frac{1}{\mathbf{e}(\mathcal{N}Q_{\Gamma_2})} \right\}. \end{aligned} \quad (7.10)$$

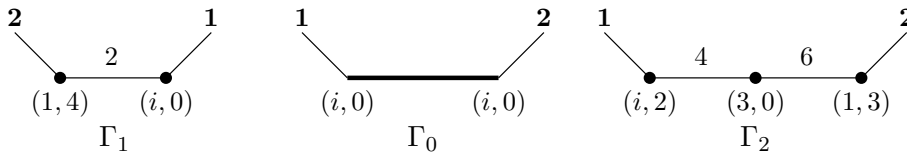


Figure 6: The three sub-strands of the strand in Figure 5

This identity remains valid with $|\Gamma_1|=0$ and/or $|\Gamma_2|=0$ if we set the corresponding integral to 1.

We now sum up (7.10) over all $\Gamma \in \mathcal{A}_i$. This is the same as summing over all pairs (Γ_1, Γ_2) of decorated strands such that

- (1) Γ_1 is a 2-pointed strand of degree $d_1 \geq 0$ such that the marked point 1 is attached to the vertex labeled i ;
- (2) Γ_2 is a 2-pointed strand of degree $d_2 \geq 0$ such that the marked point 1 is attached to the vertex labeled i .

By the localization formula (3.8),

$$\begin{aligned}
1 + \sum_{\Gamma_1} (qe^{\hbar z})^{|\Gamma_1|} & \left\{ \int_{Q_{\Gamma_1}} \frac{\mathbf{e}(\mathcal{V}'_{n;\mathbf{a}})^{(|\Gamma_1|)} \text{ev}_1^* \phi_i}{\hbar - \psi_1} \Big|_{Q_{\Gamma_1}} \frac{1}{\mathbf{e}(\mathcal{N}Q_{\Gamma_1})} \right\} \\
& = 1 + \sum_{d=1}^{\infty} (qe^{\hbar z})^d \int_{\overline{Q}_{0,2}(\mathbb{P}^{n-1}, d)} \frac{\mathbf{e}(\mathcal{V}'_{n;\mathbf{a}})^{(d)} \text{ev}_1^* \phi_i}{\hbar - \psi_1} = \mathcal{Z}_{n;\mathbf{a}}(\alpha_i, \hbar, qe^{\hbar z}); \\
1 + \sum_{\Gamma_2} q^{|\Gamma_2|} & \left\{ \int_{Q_{\Gamma_2}} \frac{\mathbf{e}(\mathcal{V}'_{n;\mathbf{a}})^{(|\Gamma_2|)} \text{ev}_1^* \phi_i}{(-\hbar) - \psi_1} \Big|_{Q_{\Gamma_2}} \frac{1}{\mathbf{e}(\mathcal{N}Q_{\Gamma_2})} \right\} \\
& = 1 + \sum_{d=0}^{\infty} q^d \int_{\overline{Q}_{0,2}(\mathbb{P}^{n-1}, d)} \frac{\mathbf{e}(\mathcal{V}'_{n;\mathbf{a}})^{(d)} \text{ev}_1^* \phi_i}{(-\hbar) - \psi_1} = \mathcal{Z}_{n;\mathbf{a}}(\alpha_i, -\hbar, q).
\end{aligned} \tag{7.11}$$

Finally, by (3.8), (7.10), and (7.11),

$$\begin{aligned}
\sum_{d=0}^{\infty} q^d \int_{\mathcal{X}'_d} e^{(\theta^* \Omega)z} F^* \mathbf{e}(\mathcal{V}_{n;\mathbf{a}}^{(d)}) & = \sum_{i=1}^{i=n} \frac{\langle \mathbf{a} \rangle \alpha_i^{\ell(\mathbf{a})} e^{\alpha_i z}}{\prod_{k \neq i} (\alpha_i - \alpha_k)} \mathcal{Z}_{n;\mathbf{a}}(\alpha_i, \hbar, qe^{\hbar z}) \mathcal{Z}_{n;\mathbf{a}}(\alpha_i, -\hbar, q) \\
& = \Phi_{\mathcal{Z}_{n;\mathbf{a}}}(\hbar, z, q),
\end{aligned}$$

as claimed in (7.5).

In the case of products of projective spaces and concavex sheaves (1.7), the spaces

$$\overline{Q}_{0,2}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1, d)) \quad \text{and} \quad \overline{\mathfrak{X}}_d = \mathbb{P}(\mathbb{C}^n \otimes \text{Sym}^d V^*)$$

are replaced by

$$\overline{Q}_{0,2}(\mathbb{P}V \times \mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_p-1}, (1, d_1, \dots, d_p)) \quad \text{and} \quad \mathbb{P}(\mathbb{C}^{n_1} \otimes \text{Sym}^{d_1} V^*) \times \dots \times \mathbb{P}(\mathbb{C}^{n_p} \otimes \text{Sym}^{d_p} V^*),$$

respectively. Lemma 7.2 then becomes

$$\Phi_{\mathcal{Z}_{n_1, \dots, n_p; \mathbf{a}}}(\hbar, z_1, \dots, z_p, q_1, \dots, q_p) = \sum_{d_1, \dots, d_p \geq 0} q_1^{d_1} \dots q_p^{d_p} \int_{\mathcal{X}'_{d_1, \dots, d_p}} e^{(\theta^* \Omega_1)z_1 + \dots + (\theta^* \Omega_p)z_p} \pi_1^* \mathbf{e}(\mathcal{V}_{n_1, \dots, n_p; \mathbf{a}}^{(d_1, \dots, d_p)}).$$

The vertices of the thick edge in Figure 5 are now labeled by a tuple (i_1, \dots, i_p) with $i_s \in [n_s]$, as needed for the extension of (5.5) described at the end of Section 5. The relation (7.9) becomes

$$\theta^* \Omega_s|_{Q_{\Gamma}} = \alpha_{s; i_s} + |\Gamma_1|_s \hbar,$$

where $|\Gamma_1|_s$ is the sum of the s -th components of the values of \mathfrak{d} on the vertices and edges of Γ_1 (corresponding to the degree of the maps to \mathbb{P}^{n_s-1}). Otherwise, the proof is identical.

8 Proof of Theorems 3 and 4

This section concludes the proof of Theorem 3 stated in Section 4. Sections 5-7 reduce this theorem to conditions on the power series $\mathcal{Y}_{n;\mathbf{a}}$ defined in (4.2); see Lemma 8.2. Based on qualitative, primarily algebraic, considerations, we show in the proof of Proposition 8.3 that this power series does indeed satisfy these conditions and thus establish Theorem 3. The only geometric considerations entering the proof of Proposition 8.3 concern moduli spaces of stable curves $\overline{\mathcal{M}}_{0,2|d}$, not moduli spaces of stable quotients $\overline{\mathcal{Q}}_{0,2}(\mathbb{P}^{n-1}, d)$. We conclude this section by showing that these conditions on $\mathcal{Y}_{n;\mathbf{a}}$ determine certain integrals on $\overline{\mathcal{M}}_{0,2|d}$ and finish the proof of Theorem 4 stated in Section 4.

Corollary 8.1. *Let $l \in \mathbb{Z}^{\geq 0}$, $n \in \mathbb{Z}^+$ and $\mathbf{a} \in (\mathbb{Z}^*)^l$. If $|\mathbf{a}| \leq n-2$,*

$$\mathcal{Z}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q) = \mathcal{Y}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q) \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[[\hbar^{-1}, q]].$$

Proof. Both sides of this identity are \mathfrak{C} -recursive and satisfy the self-polynomiality condition (no matter what n and \mathbf{a} are); see Lemma 5.4 and Propositions 6.1 and 7.1. It is immediate from (4.2) that

$$\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q) \cong 1 \pmod{\hbar^{-2}},$$

whenever $|\mathbf{a}| \leq n-2$. If in addition $d \in \mathbb{Z}^+$,

$$\dim \overline{\mathcal{Q}}_{0,2}(\mathbb{P}^{n-1}, d) - \text{rk} \mathcal{V}'_{n;\mathbf{a}}(d) = (n-|\mathbf{a}|)d + (n-2) > n-1 = \dim \mathbb{P}^{n-1}.$$

Thus,

$$\mathcal{Z}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q) \cong 1 \pmod{\hbar^{-2}},$$

whenever $|\mathbf{a}| \leq n-2$. The claim now follows from Proposition 5.3. \square

Lemma 8.2. *If $l \in \mathbb{Z}^{\geq 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$ are such that $|\mathbf{a}| \leq n$, then*

$$\mathcal{Z}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q) = \frac{\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q)}{I_{n;\mathbf{a}}(q)} \in (H_{\mathbb{T}}^*(\mathbb{P}^{n-1}))[[\hbar^{-1}, q]] \quad (8.1)$$

if and only if

$$\begin{aligned} & \mathfrak{R}_{\hbar=0} \left\{ \hbar^r \mathcal{Y}_{n;\mathbf{a}}(\alpha_i, \hbar, q) \right\} \\ &= \sum_{d=1}^{\infty} \frac{q^d}{d!} \sum_{b=0}^{d-1-r} \left(\left(\int_{\overline{\mathcal{M}}_{0,2|d}} \frac{\mathbf{e}(\mathcal{V}'_{\mathbf{a};d}(\alpha_i)) \psi_1^r \psi_2^b}{\prod_{k \neq i} \mathbf{e}(\mathcal{V}'_{1;d}(\alpha_i - \alpha_k))} \right) \mathfrak{R}_{\hbar=0} \left\{ \frac{(-1)^b}{\hbar^{b+1}} \mathcal{Y}_{n;\mathbf{a}}(\alpha_i, \hbar, q) \right\} \right) \end{aligned} \quad (8.2)$$

for all $i \in [n]$ and $r \in \mathbb{Z}^{\geq 0}$.

Proof. Since both sides of (8.1) are \mathfrak{C} -recursive (see Lemma 5.4 and Proposition 6.1) and have the same q^0 -coefficients, (8.1) holds if and only if the secondary coefficients $\mathcal{Y}_i^r(d)$ and $\mathcal{Z}_i^r(d)$ (instead of $\mathcal{F}_i^r(d)$) in the recursions (5.4) for $\mathcal{Y}_{n;\mathbf{a}}$ and $\mathcal{Z}_{n;\mathbf{a}}$ are the same. Since Proposition 6.1 describes the coefficients $\mathcal{Z}_i^r(d)$ recursively on d , (8.1) holds if and only if the coefficients $\mathcal{Y}_i^r(d)$ satisfy the same description. By Lemma 5.4 and Proposition 6.1, this is the case if and only if (8.2) holds. \square

Proposition 8.3. *If $l \in \mathbb{Z}^{\geq 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$, then*

$$\begin{aligned} & \Re_{\hbar=0} \left\{ \hbar^r \mathcal{Y}_{n;\mathbf{a}}(\alpha_i, \hbar, q) \right\} \\ &= \sum_{d=1}^{\infty} \frac{q^d}{d!} \sum_{b=0}^{d-1-r} \left(\left(\int_{\overline{\mathcal{M}}_{0,2|d}} \frac{\mathbf{e}(\mathcal{V}'_{\mathbf{a};d}(\alpha_i)) \psi_1^r \psi_2^b}{\prod_{k \neq i} \mathbf{e}(\mathcal{V}'_{1;d}(\alpha_i - \alpha_k))} \right) \Re_{\hbar=0} \left\{ \frac{(-1)^b}{\hbar^{b+1}} \mathcal{Y}_{n;\mathbf{a}}(\alpha_i, \hbar, q) \right\} \right) \end{aligned} \quad (8.3)$$

for all $i \in [n]$ and $r \in \mathbb{Z}^{\geq 0}$.

Proof. (1) Whenever $d \in \mathbb{Z}^+$ and $s \in [d]$, where $[d] = \{1, \dots, d\}$ as before, let

$$\Delta_s = \sum_{t=s+1}^d \Delta_{st} \in H^2(\overline{\mathcal{M}}_{0,2|d})$$

with $\Delta_{st} \equiv \Delta_{\{s,t\}}$ as in (2.9). For each $a_k > 0$, $s \in [d]$, and $r \in [a_k]$, there is a short exact sequence

$$\begin{aligned} 0 \longrightarrow R^0 \pi_* \mathcal{O} \left((r-1) \hat{\sigma}_s + \sum_{t=s+1}^d a_k \hat{\sigma}_t - \sigma_1 \right) &\longrightarrow R^0 \pi_* \mathcal{O} \left(r \hat{\sigma}_s + \sum_{t=s+1}^d a_k \hat{\sigma}_t - \sigma_1 \right) \\ &\longrightarrow R^0 \pi_* \mathcal{O} \left(\left(r \hat{\sigma}_s + \sum_{t=s+1}^d a_k \hat{\sigma}_t - \sigma_1 \right) \Big|_{\hat{\sigma}_s} \right) \longrightarrow 0. \end{aligned}$$

This and the second statement of Corollary 2.5 give

$$\begin{aligned} a_k > 0 \implies \mathbf{e}(\mathcal{V}'_{a_k;d}(\alpha_i)) &= \prod_{s=1}^d \prod_{r=1}^{a_k} (a_k \alpha_i - r \hat{\psi}_s + a_k \Delta_s) \\ &= a_k^{a_k d} \alpha_i^{a_k d} \prod_{s=1}^d (1 + \alpha_i^{-1} \Delta_s)^{a_k} \in H_{\mathbb{T}}^* \otimes_{\mathbb{Q}} \tilde{H}(\overline{\mathcal{M}}_{0,2|d}). \end{aligned} \quad (8.4)$$

For each $a_k < 0$, $s \in [d]$, and $r = -a_k + 1, -a_k + 2, \dots, 0$, there is a short exact sequence

$$\begin{aligned} 0 \longrightarrow R^0 \pi_* \mathcal{O} \left(\left(r \hat{\sigma}_s + \sum_{t=s+1}^d a_k \hat{\sigma}_t - \sigma_1 \right) \Big|_{\hat{\sigma}_s} \right) &\longrightarrow R^1 \pi_* \mathcal{O} \left((r-1) \hat{\sigma}_s + \sum_{t=s+1}^d a_k \hat{\sigma}_t - \sigma_1 \right) \\ &\longrightarrow R^1 \pi_* \mathcal{O} \left(r \hat{\sigma}_s + \sum_{t=s+1}^d a_k \hat{\sigma}_t - \sigma_1 \right) \longrightarrow 0. \end{aligned}$$

This and the second statement of Corollary 2.5 give

$$\begin{aligned} a_k < 0 \implies \mathbf{e}(\mathcal{V}'_{a_k;d}(\alpha_i)) &= \prod_{s=1}^d \prod_{r=0}^{-a_k-1} (a_k \alpha_i - r \hat{\psi}_s + a_k \Delta_s) \\ &= a_k^{-a_k d} \alpha_i^{-a_k d} \prod_{s=1}^d (1 + \alpha_i^{-1} \Delta_s)^{-a_k} \in H_{\mathbb{T}}^* \otimes_{\mathbb{Q}} \tilde{H}(\overline{\mathcal{M}}_{0,2|d}). \end{aligned} \quad (8.5)$$

Similarly to (8.4),

$$\mathbf{e}(\mathcal{V}'_{1;d}(\alpha_i - \alpha_k)) = (\alpha_i - \alpha_k)^d \prod_{s=1}^d (1 + (\alpha_i - \alpha_k)^{-1} \Delta_s) \in H_{\mathbb{T}}^* \otimes_{\mathbb{Q}} \tilde{H}(\overline{\mathcal{M}}_{0,2|d}). \quad (8.6)$$

(2) We denote by s_1, s_2, \dots the elementary symmetric polynomials in β_1, β_2, \dots (for any given number of formal variables β_i). For any $d \in \mathbb{Z}^+$, let

$$\mathcal{P}_2^*(d) = \{\mathbf{J} \equiv \{J_1, \dots, J_s\} : J_t \subset [d], |J_t| \geq 2, J_{t_1} \cap J_{t_2} = \emptyset \ \forall t_1 \neq t_2\}$$

be the collection of disjoint subsets of $\{1, \dots, d\}$ of cardinality at least 2 (including the empty collection). For each $\mathbf{J} \in \mathcal{P}_2^*(d)$, let

$$\Delta_{\mathbf{J}} = \Delta_{J_1} \cdots \Delta_{J_s} \in \tilde{H}^*(\overline{\mathcal{M}}_{0,2|d}) \quad \text{if } \mathbf{J} \equiv \{J_1, \dots, J_s\}, J_{t_1} \neq J_{t_2} \ \forall t_1 \neq t_2.$$

By the first statement of Corollary 2.5, there exist polynomials $\mathcal{H}_{d,\mathbf{J}}$, dependent only on $d \in \mathbb{Z}^+$ and $\mathbf{J} \in \mathcal{P}_2^*(d)$, but not on n , such that

$$\frac{1}{\prod_{s=1}^d \prod_{k=1}^{n-1} (1 + \beta_k \Delta_s)} = \sum_{\mathbf{J} \in \mathcal{P}_2^*(d)} \mathcal{H}_{d,\mathbf{J}}(s_1, \dots, s_{d-1}) \Delta_{\mathbf{J}} \in \mathbb{Q}[\beta_1, \dots, \beta_{n-1}] \otimes_{\mathbb{Q}} \tilde{H}^*(\overline{\mathcal{M}}_{0,2|d}).^9$$

Thus, there exists a polynomial $\mathcal{H}_{\mathbf{a};d}^{r,b}$, independent of n , such that

$$\frac{(-1)^b}{d!} \int_{\overline{\mathcal{M}}_{0,2|d}} \frac{\prod_{s=1}^d (1 + y \Delta_s)^{|\mathbf{a}|} \psi_1^r \psi_2^b}{\prod_{s=1}^d \prod_{k=1}^{n-1} (1 + \beta_k \Delta_s)} = \mathcal{H}_{\mathbf{a};d}^{r,b}(y, s_1, \dots, s_{d-1}) \in \mathbb{Q}[y, \beta_1, \dots, \beta_{n-1}]. \quad (8.7)$$

Similarly, for any $d, d' \in \mathbb{Z}^{\geq 0}$ there exists a polynomial $\mathcal{Y}_{\mathbf{a};d,d'}$, independent of n , such that

$$\left[\frac{\prod_{a_k > 0} \prod_{r=1}^{a_k d} (1 + r y^{-1} \hbar) \prod_{a_k < 0} \prod_{r=0}^{-a_k d - 1} (1 - r y^{-1} \hbar)}{d! \prod_{r=1}^d \prod_{k=1}^{n-1} (1 + r \beta_k^{-1} \hbar)} \right]_{\hbar; d'} = \mathcal{Y}_{\mathbf{a};d,d'}(y, s_1, \dots, s_{d'}). \quad (8.8)$$

(3) By (4.2) and (8.4)-(8.8), (8.3) is equivalent to

$$\mathcal{Y}_{\mathbf{a};d,d-1-r}(y, s_1, s_2, \dots) = \sum_{\substack{d_1 + d_2 = d \\ d_1 \geq 1}} \sum_{b=0}^{d_1 - 1 - r} \mathcal{H}_{\mathbf{a};d_1}^{r,b}(y, s_1, s_2, \dots) \mathcal{Y}_{\mathbf{a};d_2,d_2+b}(y, s_1, s_2, \dots) \quad \forall d \in \mathbb{Z}^+; \quad (8.9)$$

this is obtained by taking the coefficients of q^d of the two sides of (8.3), factoring out

$$\frac{\prod_{a_k > 0} (a_k^{a_k d} \alpha_i^{a_k d}) \prod_{a_k < 0} (a_k^{-a_k d} \alpha_i^{-a_k d})}{\prod_{k \neq i} (\alpha_i - \alpha_k)^d},$$

⁹Whatever polynomials work for $n \geq d$ work for all n ; this can be seen by setting the extra β_k 's to 0.

replacing α_i^{-1} by y and $\{(\alpha_i - \alpha_k)^{-1} : k \neq i\}$ by $\{\beta_1, \dots, \beta_{n-1}\}$. By Lemma 8.2 and Corollary 8.1, (8.9) holds whenever $|\mathbf{a}| \leq n-2$. Since (8.9) does not involve n , it holds for all \mathbf{a} . Thus, (8.3) holds for all pairs (n, \mathbf{a}) . \square

In the case of products of projective spaces and concavex sheaves (1.7), α_i and q in (8.2) and (8.3) are replaced by $(\alpha_{i_1}, \dots, \alpha_{i_p})$ with $i_s \in [n_s]$ and (q_1, \dots, q_p) with the right-hand sides modified as in (6.13). In the proof of Proposition 8.3, we then obtain relations between elementary symmetric polynomials in

$$\{\alpha_{1;1}, \dots, \alpha_{1;n_1}\}, \quad \dots \quad \{\alpha_{p;1}, \dots, \alpha_{p;n_p}\}$$

that depend on \mathbf{a} , but not on n_1, \dots, n_p . They again hold if $|\mathbf{a}| \leq n_1 + \dots + n_p - 2$ and thus in all cases.

Proof of Theorem 4. For each $d \in \mathbb{Z}^+$, denote by $D_{1\hat{1};2} \subset \overline{\mathcal{M}}_{0,2|d}$ the divisor whose general element is a two-component rational curve, with one of the components carrying the marked point 1 and the fleck $\hat{1}$ and the other component carrying the marked point 2. Since the second component must then carry at least one of the remaining flecks, there is a birational isomorphism

$$D_{1\hat{1};2} \approx \bigsqcup_{\emptyset \neq I \subset \{2, \dots, d\}} \overline{\mathcal{M}}_{0,2|(d-|I|)} \times \overline{\mathcal{M}}_{0,2||I|}. \quad (8.10)$$

If π_1, π_2 are the two component projection maps,

$$\begin{aligned} \psi_i|_{D_{1\hat{1};2}} &\approx \pi_i^* \psi_i \quad i = 1, 2, \\ \mathbf{e}(\mathcal{V}'_{\mathbf{a};d}(\beta))|_{\overline{\mathcal{M}}_{0,2|(d-|I|)} \times \overline{\mathcal{M}}_{0,2||I|}} &\approx \pi_1^* \mathbf{e}(\mathcal{V}'_{\mathbf{a};d-|I|}(\beta)) \cdot \pi_2^* \mathbf{e}(\mathcal{V}'_{\mathbf{a};|I|}(\beta)). \end{aligned} \quad (8.11)$$

On the other hand, by the first identity in (2.11) and induction on d ,

$$\psi_2 = D_{1\hat{1};2} \in H^2(\overline{\mathcal{M}}_{0,2|d}). \quad (8.12)$$

By (8.10)-(8.12),

$$\begin{aligned} \int_{\overline{\mathcal{M}}_{0,2|d}} \frac{\mathbf{e}(\mathcal{V}'_{\mathbf{a};d}(\alpha_i)) \psi_1^{b_1} \psi_2^{b_2}}{\prod_{k \neq i} \mathbf{e}(\mathcal{V}'_{1;d}(\alpha_i - \alpha_k))} &= \int_{D_{1\hat{1};2}} \frac{\mathbf{e}(\mathcal{V}'_{\mathbf{a};d}(\alpha_i)) \psi_1^{b_1} \psi_2^{b_2-1}}{\prod_{k \neq i} \mathbf{e}(\mathcal{V}'_{1;d}(\alpha_i - \alpha_k))} \\ &= \sum_{\substack{d_1, d_2 \geq 1 \\ d_1 + d_2 = d}} \binom{d-1}{d_1-1} \left(\int_{\overline{\mathcal{M}}_{0,2|d_1}} \frac{\mathbf{e}(\mathcal{V}'_{\mathbf{a};d_1}(\alpha_i)) \psi_1^{b_1}}{\prod_{k \neq i} \mathbf{e}(\mathcal{V}'_{1;d_1}(\alpha_i - \alpha_k))} \right) \left(\int_{\overline{\mathcal{M}}_{0,2|d_2}} \frac{\mathbf{e}(\mathcal{V}'_{\mathbf{a};d_2}(\alpha_i)) \psi_2^{b_2-1}}{\prod_{k \neq i} \mathbf{e}(\mathcal{V}'_{1;d_2}(\alpha_i - \alpha_k))} \right) \end{aligned} \quad (8.13)$$

whenever $b_2 \in \mathbb{Z}^+$. For any $b_1, b_2 \in \mathbb{Z}^{\geq 0}$, let

$$\mathcal{F}_{b_1, b_2}(q) = \sum_{d=1}^{\infty} \frac{q^d}{d!} \int_{\overline{\mathcal{M}}_{0,2|d}} \frac{\mathbf{e}(\mathcal{V}'_{\mathbf{a};d}(\alpha_i)) \psi_1^{b_1} \psi_2^{b_2}}{\prod_{k \neq i} \mathbf{e}(\mathcal{V}'_{1;d}(\alpha_i - \alpha_k))} \in q\mathbb{Q}_{\alpha}[[q]].$$

By (8.13),

$$\mathcal{F}'_{b_1, b_2}(q) = \mathcal{F}'_{b_1, 0}(q) \cdot \mathcal{F}_{0, b_2-1}(q) \quad \forall b_2 \in \mathbb{Z}^+, \quad (8.14)$$

where $\mathcal{F}' \equiv q \frac{d}{dq} \mathcal{F}$. By induction on b_2 , this gives

$$\mathcal{F}_{0,b_2}(q) = \frac{1}{(b_2+1)!} \mathcal{F}_{0,0}(q)^{b_2+1}.$$

Combining this with (8.14) and using symmetry, we obtain

$$\begin{aligned} \mathcal{F}'_{b_1,b_2}(q) &= \frac{1}{b_1!} \mathcal{F}_{0,0}(q)^{b_1} \mathcal{F}'_{0,0}(q) \cdot \frac{1}{b_2!} \mathcal{F}_{0,0}(q)^{b_2} \implies \\ \mathcal{F}_{b_1,b_2}(q) &= \frac{1}{(b_1+b_2+1)!} \binom{b_1+b_2}{b_1} \mathcal{F}_{0,0}(q)^{b_1+b_2+1}. \end{aligned} \quad (8.15)$$

Thus, the $r=0$ case of (8.3) is equivalent to

$$\Re_{\hbar=0} \left\{ e^{-\frac{\mathcal{F}_{0,0}(q)}{\hbar}} \mathcal{Y}_{n;\mathbf{a}}(\alpha_i, \hbar, q) \right\} = 0. \quad (8.16)$$

By [16, Section 2.1], this relation determines $\mathcal{F}_{0,0}(q) \in q\mathbb{Q}_\alpha[[q]]$ uniquely. Thus, by [18, Remark 4.5], $\mathcal{F}_{0,0}(q) = \xi_{n;\mathbf{a}}(\alpha_i; q)$. It follows that (8.15) is equivalent to the identity in Theorem 4. \square

Remark 8.4. By (8.15), for any $r^* \in \mathbb{Z}^{\geq 0}$ the set of equations (8.3) with $r = 0, 1, \dots, r^*$ is an invertible linear combination of the set of relations

$$\Re_{\hbar=0} \left\{ \hbar^r e^{-\frac{\mathcal{F}_{0,0}(q)}{\hbar}} \mathcal{Y}_{n;\mathbf{a}}(\alpha_i, \hbar, q) \right\} = 0 \quad r = 0, 1, \dots, r^*.$$

Thus, by (8.15), the statement of Proposition 8.3 is equivalent to the condition that the coefficients of the power series

$$e^{-\mathcal{F}_{0,0}(q)/\hbar} \mathcal{Y}_{n;\mathbf{a}}(\alpha_i, \hbar, q) \in \mathbb{Q}_\alpha(\hbar)[[q]]$$

are regular at $\hbar=0$. This is indeed the case for $\mathcal{F}_{0,0}(q) = \xi_{n;\mathbf{a}}(\alpha_i; q)$ by [18, Remark 4.5].

Remark 8.5. The above approach can be used to eliminate ψ -classes from twisted integrals over $\overline{\mathcal{M}}_{0,m|d}$ with $m \geq 3$. For example, let

$$\mathcal{F}_{b_1,b_2,b_3}(q) = \sum_{d=0}^{\infty} \frac{q^d}{d!} \int_{\overline{\mathcal{M}}_{0,3|d}} \frac{\mathbf{e}(\mathcal{V}'_{\mathbf{a};d}(\alpha_i)) \psi_1^{b_1} \psi_2^{b_2} \psi_3^{b_3}}{\prod_{k \neq i} \mathbf{e}(\mathcal{V}'_{1;d}(\alpha_i - \alpha_k))}.$$

Using $\psi_3 = D_{12;3}$ on $\overline{\mathcal{M}}_{0,3|d}$, we find that

$$\mathcal{F}_{b_1,b_2,b_3}(q) = \mathcal{F}_{b_1,b_2,0}(q) \cdot \mathcal{F}_{0,b_3-1}(q) \quad \forall b_3 \in \mathbb{Z}^+ \implies \mathcal{F}_{b_1,b_2,b_3}(q) = \frac{\xi_{n;\mathbf{a}}(\alpha_i, q)^{b_1+b_2+b_3}}{b_1!b_2!b_3!} \mathcal{F}_{0,0,0}(q).$$

Multiplying the last equation by $\hbar_1^{-b_1-1} \hbar_2^{-b_2-1} \hbar_3^{-b_3-1}$ and summing over $b_1, b_2, b_3 \geq 0$, we obtain

$$\begin{aligned} \sum_{d=0}^{\infty} \frac{q^d}{d!} \int_{\overline{\mathcal{M}}_{0,3|d}} \frac{\mathbf{e}(\mathcal{V}'_{\mathbf{a};d}(\alpha_i))}{\prod_{k \neq i} \mathbf{e}(\mathcal{V}'_{1;d}(\alpha_i - \alpha_k)) (\hbar_1 - \psi_1)(\hbar_2 - \psi_2)(\hbar_3 - \psi_3)} \\ = \frac{1}{\hbar_1 \hbar_2 \hbar_3} e^{\frac{\xi_{n;\mathbf{a}}(\alpha_i, q)}{\hbar_1} + \frac{\xi_{n;\mathbf{a}}(\alpha_i, q)}{\hbar_2} + \frac{\xi_{n;\mathbf{a}}(\alpha_i, q)}{\hbar_3}} \mathcal{F}_{0,0,0}(q) \in \mathbb{Q}_\alpha[[\hbar_1^{-1}, \hbar_2^{-1}, \hbar_3^{-1}, q]]. \end{aligned}$$

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